DISMA Dipartimento di Scienze Matematiche G. L. Lagrange

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Critical sets of PL and discrete Morse theory: a correspondence

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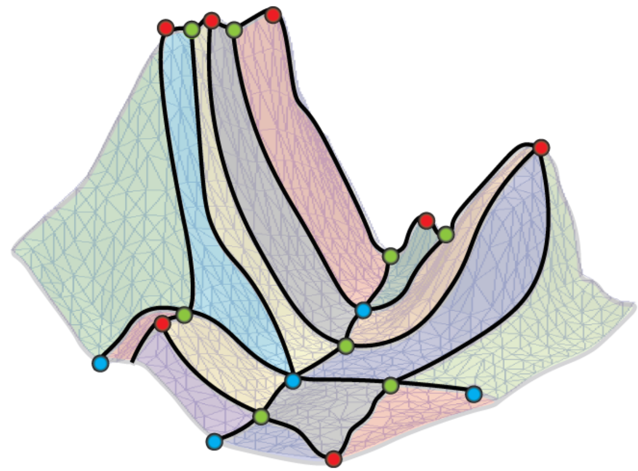
Motivation

Morse Theory:

Powerful ***topological tool*** for efficiently analyzing a ***manifold*** by studying the behavior of a smooth ***scalar function*** defined on it

Effective ***applications*** in

- *Data Segmentation*
- *Homology Computation*
- *Multi-Resolution Analysis*



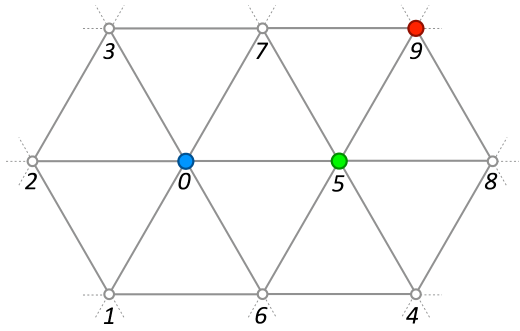
Two ***discretized*** versions of Morse theory gained a prominent role in the literature:

Piecewise-Linear (PL) Morse Theory
[Banchoff 1967]

Discrete Morse Theory
[Forman 1998]

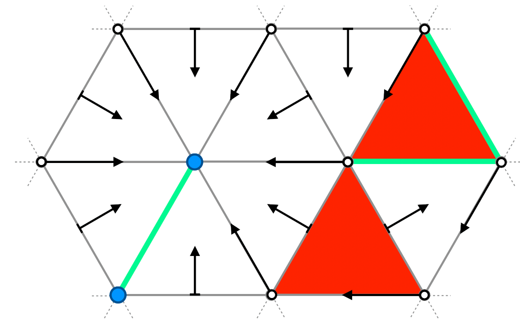
Motivation

PL Morse Theory



- Defined for ***manifold*** domains
- Need for a ***scalar function*** defined on the vertices
- ***Critical points lay on vertices***
- Approach close to ***common intuition***

Discrete Morse Theory



- Defined for ***arbitrary*** domains
- ***Gradient-based*** approach: no need for an explicit function
- ***Critical elements are simplices***
- ***Combinatorial*** approach

Can a correspondence between these two worlds be established?

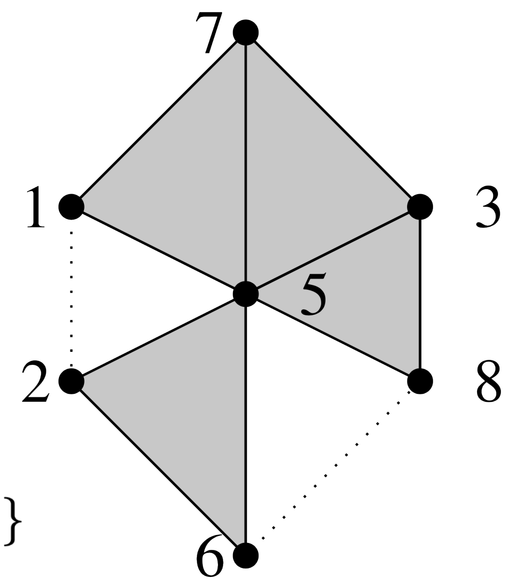
Outline

- **Piecewise-Linear Morse Theory**
  - *Equivalence between the notions of PL critical points*
- **Discrete Morse Theory**
  - *Discrete critical simplices*
- **Relating PL and Discrete Critical Sets**
  - *An explicit correspondence*
  - *Construction of RP discrete gradient vector fields*
- **Conclusions and Future Developments**

Piecewise-Linear Morse Theory

PL Critical Points [Banchoff 1967]

Let f be an *injective scalar function* defined on the vertices of a *combinatorial d-manifold* Σ with ***d = 2***



- A triangle σ := uvw has ***v middle for f*** if ***f(u) < f(v) < f(w)***
- For a vertex v, we set

$$\iota(v, f) := 1 - \frac{1}{2} \cdot \# \{\text{triangles in } star(v) \text{ with } v \text{ middle for } f\}$$

A vertex v of Σ is classified as follows:

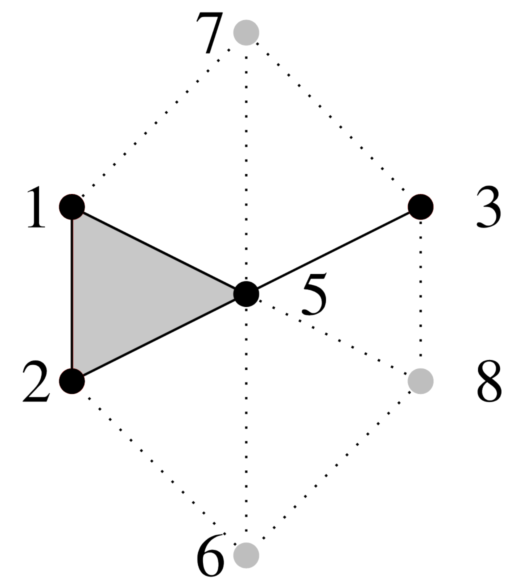
$$\iota(v, f) = \begin{cases} 1 & \leftrightarrow \quad v \text{ is a } minimum \text{ or } maximum \\ 0 & \leftrightarrow \quad v \text{ is a } regular \text{ point} \\ -k < 0 & \leftrightarrow \quad v \text{ is a } saddle \text{ of } multiplicity\ k \end{cases}$$

Piecewise-Linear Morse Theory

PL Critical Points [Edelsbrunner et al. 2001]

Let f be an *injective scalar function* defined on the vertices of a *combinatorial d-manifold* Σ with ***d = 2***

- A ***section*** of star⁻(v) is an edge or a triangle in star⁻(v)
- A collection S of sections is a ***contiguous section*** of star⁻(v) if S \ {v} is connected
- A ***wedge*** of star⁻(v) is a contiguous section of star⁻(v) whose boundary in link⁻(v) is not a cycle



Letting W the number of wedges of star⁻(v), v is classified as follows:

$$W = \begin{cases} 0 & \leftrightarrow \quad v \text{ is a } \textit{minimum} \text{ or } \textit{maximum} \\ 1 & \leftrightarrow \quad v \text{ is a } \textit{regular} \text{ point} \\ k+1>1 & \leftrightarrow \quad v \text{ is a } \textit{saddle} \text{ of } \textit{multiplicity } k \end{cases}$$

Piecewise-Linear Morse Theory

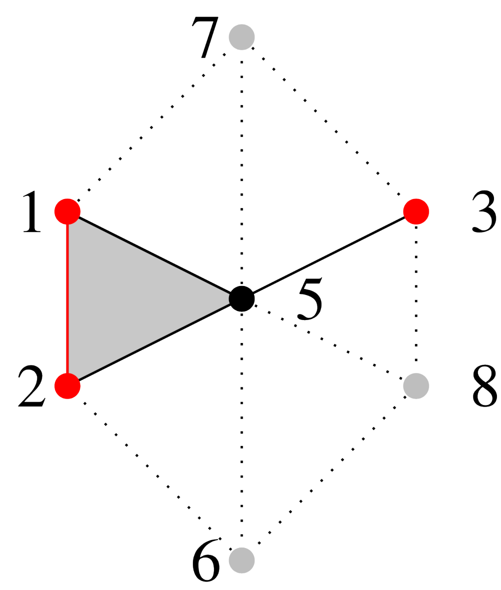
PL Critical Points [Brehm and Kühnel 1987]

Let f be an *injective scalar function* defined on the vertices of a *combinatorial d-manifold* Σ with ***arbitrary d***

A vertex v is classified as ***critical*** for f if the ***relative homology***

$$H_*(|\Sigma^\ell|, |\Sigma^\ell| \setminus \{v\})$$

is non-trivial, where $\ell = f(v)$



A critical point v has ***index i*** and ***multiplicity k_i*** if $\beta_i(|\Sigma^\ell|, |\Sigma^\ell| \setminus \{v\}) = k_i$

- Its ***total multiplicity*** is defined as $k := \Sigma_i k_i$

Vertex v is called:

- *minimum*, if v has index 0
- *maximum*, if v has index d
- *saddle*, otherwise

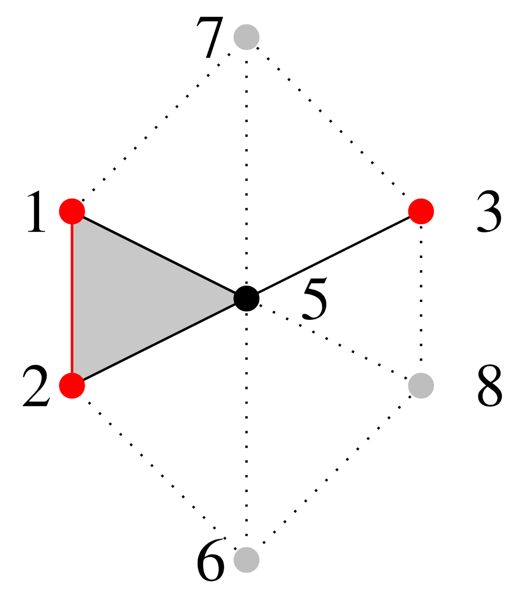
Piecewise-Linear Morse Theory

PL Critical Points [Edelsbrunner and Harer 2010]

Let f be an *injective scalar function* defined on the vertices of a *combinatorial d-manifold* Σ with ***arbitrary d***

- Let $\tilde{\beta}_j$ be the rank of the ***reduced jth homology*** group of link⁻(v)
- A vertex v of is called ***regular*** if, for any j = −1, 0, 1, ..., d,

$$\tilde{\beta}_j = 0$$



Else, v is called a ***critical*** point of ***index i*** and ***multiplicity k*** of f if

$$\tilde{\beta}_j = \begin{cases} k & \text{for } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

Piecewise-Linear Morse Theory

Equivalence between the notions of PL critical point

Theorem:

Let v be a vertex of a combinatorial manifold Σ of arbitrary dimension d endowed with an injective scalar function f defined on its vertices

*The following statements are **equivalent**:*

- *v is a **critical** point of f of index i and multiplicity k_i* ***for [Brehm and Kühnel 1987]***
- *v is a **critical** point of f of the same index and the same multiplicity* ***for [Edelsbrunner and Harer 2010]***

Moreover, for d = 2, all the previously introduced notions of critical points are equivalent

So, in the following, it will not be ambiguous to address a vertex as a PL critical point without specifying which definition we are adopting

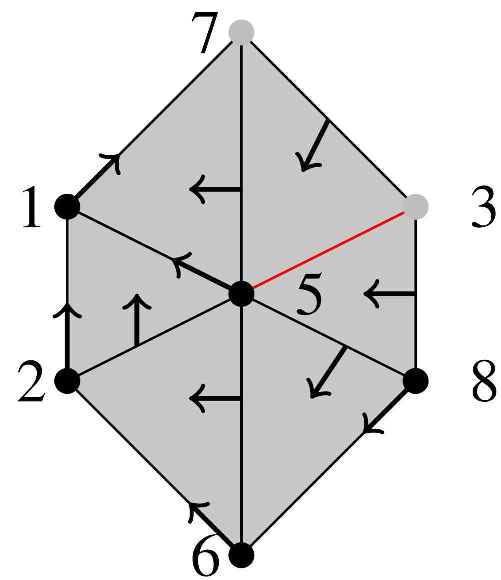
Discrete Morse Theory

Discrete Critical Simplices [Forman 1998]

Given an arbitrary simplicial complex Σ,

a ***discrete gradient vector field V*** is a collection of **pairs** in $\Sigma \times \Sigma$

- for each pair $(\sigma, \tau) \in V$, σ is a face of τ with $\dim(\sigma) = \dim(\tau) - 1$
- each simplex of Σ is in at most one pair of *V*
- free of ***closed*** V-paths



Given a discrete gradient vector field V, an i-simplex σ of Σ is called:

- ***regular***, if it belongs to a pair of V
- ***discrete critical simplex of index i***, otherwise

Relating PL and Discrete Critical Sets

Correspondence between critical sets:

Let Σ be a combinatorial manifold of arbitrary dimension d endowed with an injective scalar function f defined on its vertices and a discrete gradient vector field V

Definition:

*V is called **relatively perfect** (**RP**) with respect to f if, for any $i \in \mathbb{N}$ and any $\ell \in Im(f)$,*

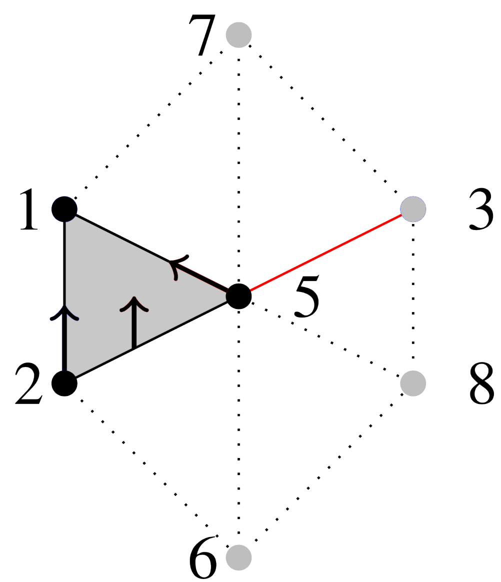
$$m_i^{\ell}(V) = \beta_i(\Sigma^{\ell}, \Sigma^{\ell'})$$

where

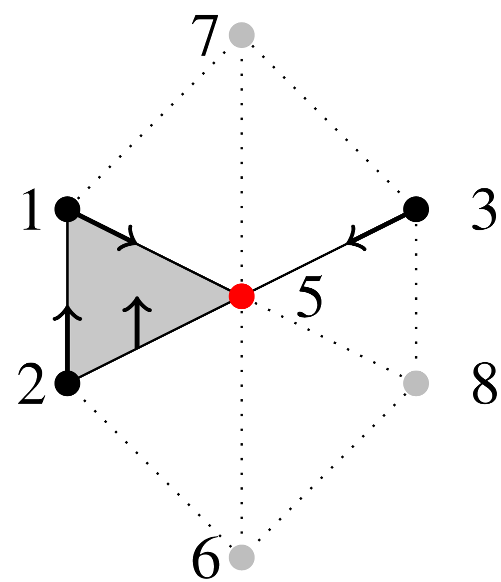
- *ℓ' denotes the greatest value in the image of f among the ones strictly lower than ℓ*
- *$m_i^{\ell}(V)$ denotes the number of discrete critical i-simplices for V in $\Sigma^{\ell} \setminus \Sigma^{\ell'}$*
- *$\beta_i(\Sigma^{\ell}, \Sigma^{\ell'})$ denotes the number of variations in the i^{th} homology group occurred at value ℓ*

Relating PL and Discrete Critical Sets

Correspondence between critical sets:



RP Gradient



Non-RP Gradient

Relating PL and Discrete Critical Sets

Correspondence between critical sets:

Let Σ be a combinatorial manifold of arbitrary dimension d endowed with an injective scalar function f defined on its vertices and a discrete gradient vector field V

Theorem:

If V is RP with respect to f then,

*a vertex v is a **PL critical** point of index i and multiplicity k_i of f*

if and only if

*there are exactly k_i **discrete critical i-simplices** σ of V such that $\sigma \in star(v)$ and $f_{max}(\sigma)=f(v)$*

Corollary:

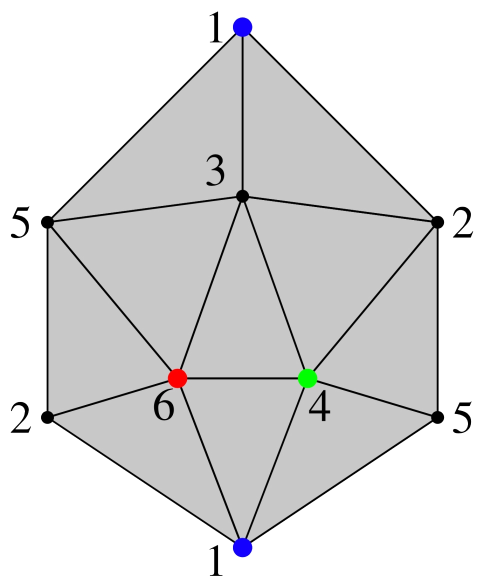
If V is RP with respect to f then,

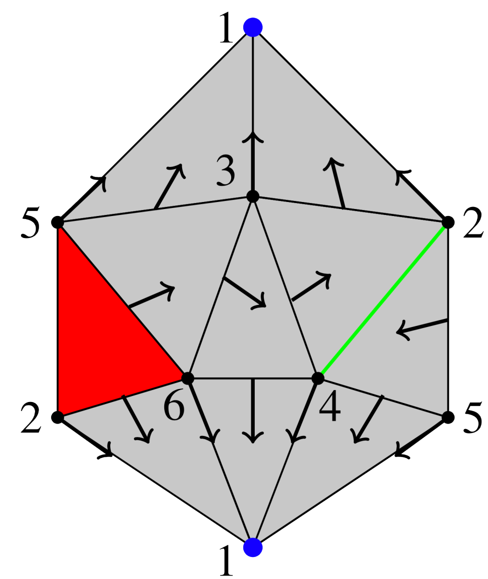
*there exists there is a **1-to-k_i correspondence** between PL critical points of index i and multiplicity k_i of f and discrete critical i-simplices σ of V such that $\sigma \in star(v)$ and $f_{max}(\sigma) = f(v)$*

*In particular, if f is PL Morse, then the correspondence is **bijective***

Relating PL and Discrete Critical Sets

Correspondence between critical sets:





PL Critical Points

Discrete Critical Simplices

Relating PL and Discrete Critical Sets

Construction of RP discrete gradient vector fields:

Let Σ be a combinatorial manifold of dimension d endowed with an injective scalar function f defined on its vertices

Theorem:

*For **d ≤ 3**, there **exists** a discrete gradient vector field V on Σ that is **RP** with respect to f*

Corollary:

*For **d ≤ 3**, there **exists** a discrete gradient vector field V on Σ (RP w.r.t. f) such that there is a **1-to-k_i correspondence** between PL critical points of index i and multiplicity k_i of f and discrete critical i-simplices σ of V such that $f_{max}(\sigma) = f(v)$*

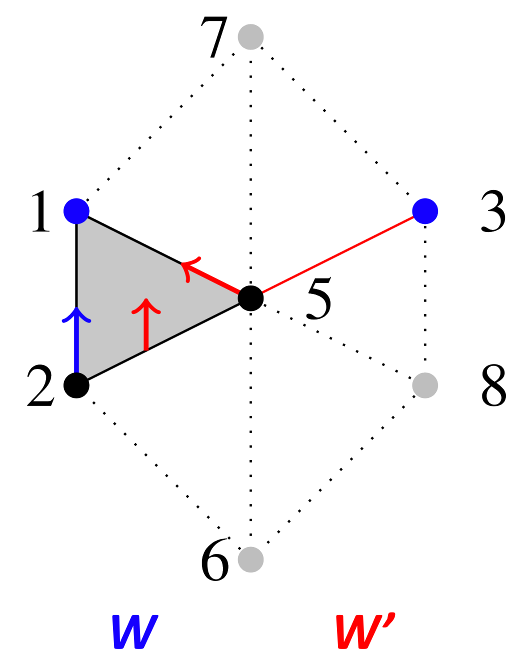
*If f is PL Morse, then the correspondence is **bijective***

Relating PL and Discrete Critical Sets

Construction of RP discrete gradient vector fields:

Sketch of the Proof:

- If $|\Gamma| \subseteq \mathbb{S}^2$, Γ admits a ***perfect*** discrete gradient vector field
- A gradient V on Σ can be constructed as ***union of gradients on star⁻(v)*** for any vertex v
- Since Σ is a ***combinatorial 3-manifold***, $|\text{link}^-(v)| \subseteq \mathbb{S}^2$, and so, ***link⁻(v) admits a perfect gradient W***
- Starting from W, ***construct a gradient W' on star⁻(v)***
  - If $(\alpha, \beta) \in$ W, then set $(v\alpha, v\beta) \in$ W'
  - If γ is a discrete critical i-simplex of W with i > 0, then set $v\gamma$ as critical for W'
  - If $\gamma_1, \ldots, \gamma_m$ are the discrete critical 0-simplices of W, then set $(v, v\gamma_1) \in$ W' and $v\gamma_2, \ldots, v\gamma_m$ as critical for W'
- Define ***V as the union of all the W'*** and prove that, by construction, ***V is RP w.r.t. f***



Conclusions and Future Developments

In Summary:

In the presented work, we have

- ***established a dimension agnostic correspondence*** between the set of PL critical points and that of discrete critical simplices
- ***improved the only previous work*** in this field by [Lewiner 2013] (limited to dimension 2 and requiring barycentric subdivisions)
- ***shown an algorithmic strategy*** for building a relatively perfect discrete gradient vector field up to dimension 3

In the near future, we plan to adopt the retrieved formal and operative connection to

- ***Morse-Smale complexes***
- ***Bifurcation theory***
- ***Steepest descent PL flows***

Thank you for the attention

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