A Kernel for Multi-Parameter Persistence

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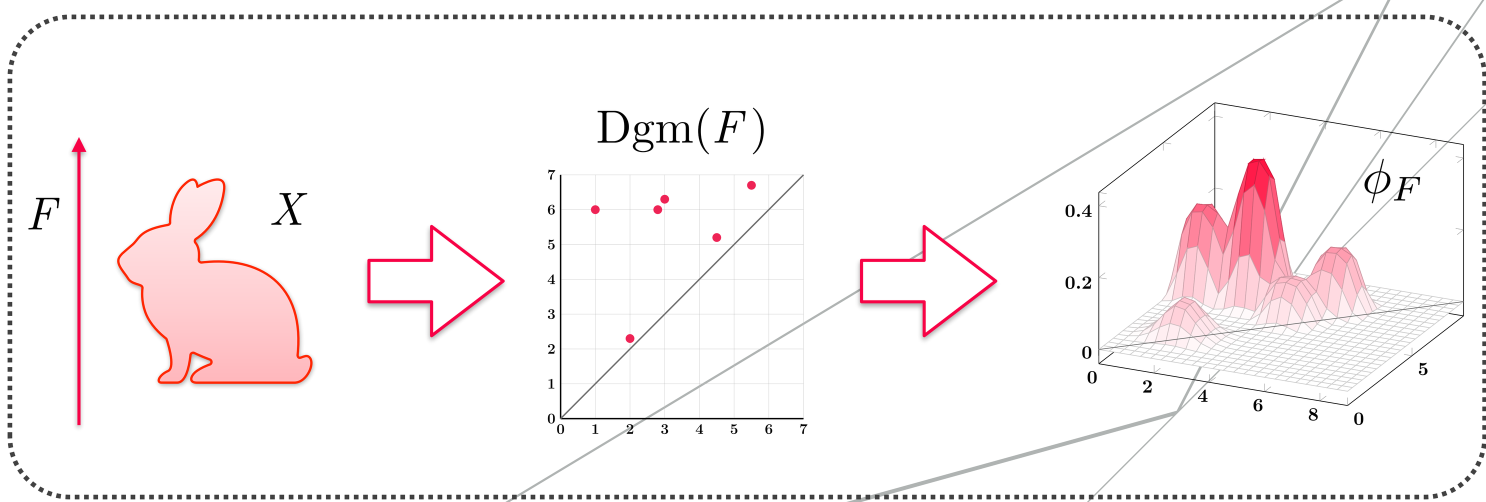
Kernel for 1-parameter Persistent Homology:

Given a topological space X filtered by a scalar function F,

a feature map ϕ transforms the persistence diagram of (X,F) into a L^2-function ϕ_F

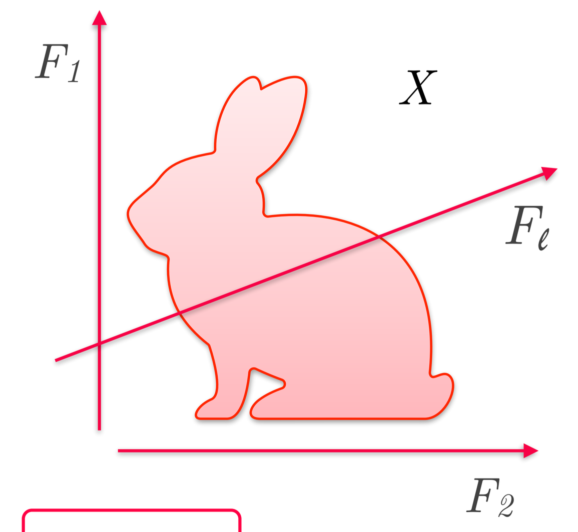


This enables the definition of a kernel $k(F,G) := \langle \phi_F, \phi_G \rangle_{L^2}$

- establishing a notion of inner product of persistence diagrams
- opening TDA to applications in statistics and machine learning

2-parameter Persistent Homology:

Given a topological space X filtered by a 2-parameter function F=(F_1,F_2),



for each line ℓ,
it is possible to associate to (X,F)
a persistence diagram $\mathrm{Dgm}(F_\ell)$
and, so, a L^2-function ϕ_{F_ℓ}

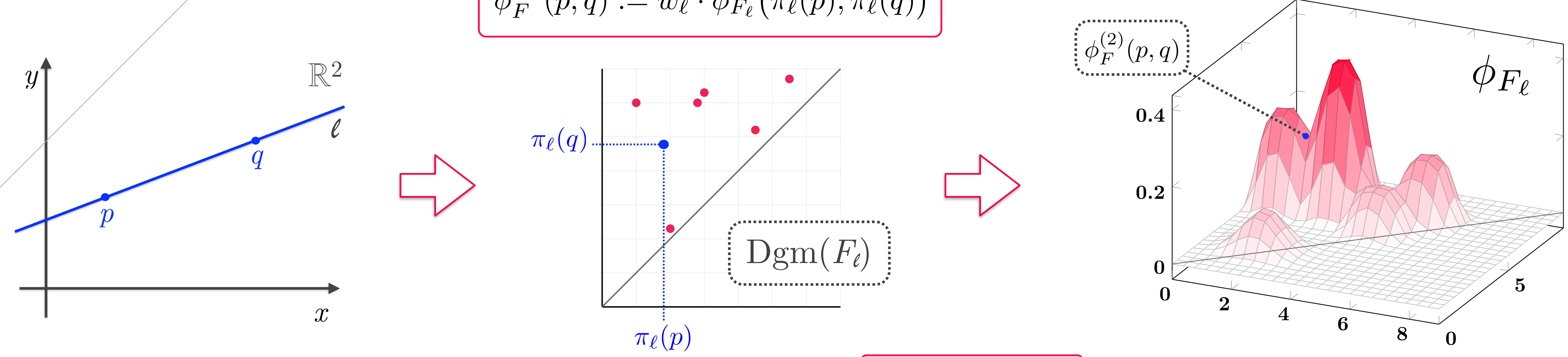
Goal:

Inspired by the matching distance [1], combine 1-parameter feature maps ϕ_{F_ℓ} to define a kernel for 2-parameter persistent homology

Feature Map for 2-parameter Persistent Homology:

Let X be a topological space and let F be a 2-parameter function filtering X. We define the feature map $\phi^{(2)}_F : \mathbb{R}^4 \longrightarrow \mathbb{R}$, for any $p,q \in \mathbb{R}^2$ with $p<q$, as

$$\phi^{(2)}_F(p,q) := w_\ell \cdot \phi_{F_\ell}\big(\pi_\ell(p), \pi_\ell(q)\big)$$



where:

- ℓ is the line determined by p and q
- $\pi_\ell(p), \pi_\ell(q)$ are the parameter values taken by p and q as points of ℓ
- w_ℓ is a weight associated to ℓ (close to 0 for almost vertical and almost horizontal lines)

Assumptions:

- p,q belong to a rectangle $R \subseteq \mathbb{R}^2$
- there exists a value N such that, for any line ℓ, $\mathrm{Dgm}(F_\ell)$ has at most N off-diagonal points

Definition:

If there exists a constant B>0,

such that, for any line ℓ, ϕ_{F_ℓ} is bounded by B, then

$\phi^{(2)}_F$ is well-defined,

i.e., $\phi^{(2)}_F \in L^2(\mathbb{R}^4)$ and $k(F,G) = \int_{\mathbb{R}^4} \phi^{(2)}_F \cdot \phi^{(2)}_G \, d\mu \in \mathbb{R}$

Stability:

If there exists a constant C> 0 such that, for any two 2-parameter filtrations F,G of a topological space X and any $p,q \in \mathbb{R}^2$ with $p<q$

$$|\phi_{F_\ell}(\pi_\ell(p), \pi_\ell(q)) - \phi_{G_\ell}(\pi_\ell(p), \pi_\ell(q))| \leq C \cdot N \cdot d_B\big(\mathrm{Dgm}(F_\ell), \mathrm{Dgm}(G_\ell)\big) \quad (\bigstar)$$

then, the kernel determined by the feature map $\phi^{(2)}$ is stable,
i.e., there exists a constant C'> 0 such that, for any two 2-parameter filtrations F,G of a topological space X,

$$\|\phi^{(2)}_F - \phi^{(2)}_G\|_{L^2} \leq C' \cdot N \cdot \mathrm{Area}(R) \cdot \|F - G\|_\infty$$

where $\|F-G\|_\infty = \sup_{x \in X} \|F(x) - G(x)\|_2$

Computability:

Let ϕ be a feature map for 1-parameter persistent homology that

- satisfies ($\bigstar$)
- is Lipschitz

Given two topological spaces filtered by 2-parameter functions (X,F), (Y,G) of size n and a value ε>0, the kernel $k(F,G)$ can be computed up to an absolute error ε

in polynomial time with respect to n and 1/ε

Genericity:

The introduced kernel enables the extension to the 2-parameter framework of several kernels for 1-parameter persistent homology:

- Persistence space-scale kernel [2]
- Persistence weighted Gaussian kernel [3]
- Persistence images [4]
- Persistence landscape [5]

The definition, the stability and the computability of the 2-parameter kernels derived from these 1-parameter kernels are achieved by the stated results

References:

[1] - S. Biasotti, A. Cerri, P. Frosini, D. Giorgi. **A new algorithm for computing the 2-dimensional matching distance between size functions**. *Pattern Recognition Letters* 32.14 (2011): 1735-1746.

[2] - J. Reininghaus, S. Huber, U. Bauer, R. Kwitt. **A stable multi-scale kernel for topological machine learning**. *Proceedings of the IEEE conference on computer vision and pattern recognition* (2015).

[3] - G. Kusano, Y. Hiraoka, K. Fukumizu. **Persistence weighted Gaussian kernel for topological data analysis**. *International Conference on Machine Learning* (2016).

[4] - H. Adams, T. Emerson, M. Kirby, R. Neville, C. Peterson, P. Shipman, S. Chepushtanova, E. Hanson, F. Motta, L. Ziegelmeier. **Persistence images: A stable vector representation of persistent homology**. *The Journal of Machine Learning Research* 18.1 (2017): 218-252.

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