STAG 2016 - Smart Tools and Apps in computer Graphics

Persistent homology: a step-by-step introduction for newcomers

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Joint work with:

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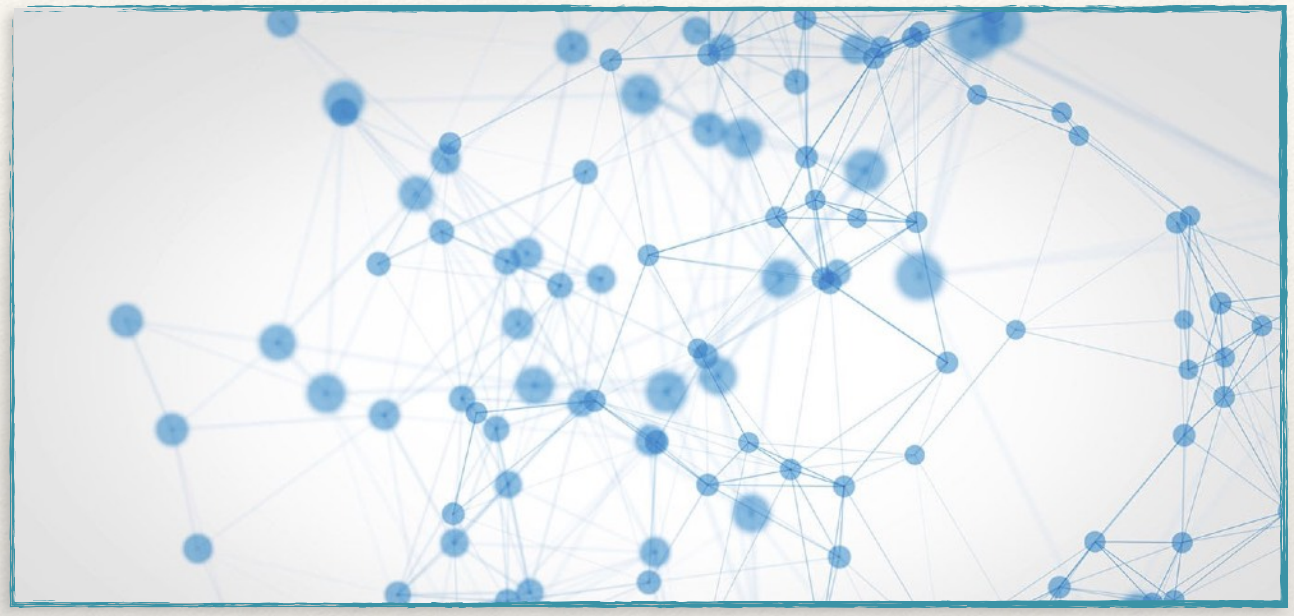
Topological Data Analysis



"Data has shape and
shape has meaning"
Gunnar Carlsson

Topological Data Analysis (TDA) is that branch of mathematics concerned with characterizing the properties of a shape

One of the most **meaningful tool** in TDA is

Persistent Homology

Persistent Homology

Persistent homology allows for **describing the changes in the shape** of an evolving object

Application domains of persistent homology:

From:

Shape Analysis

To:

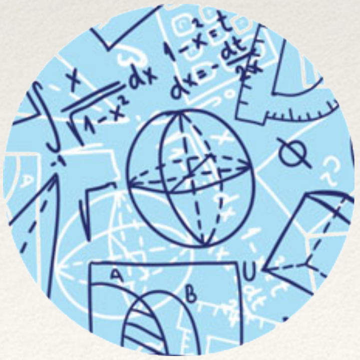
Neuroscience
Chemistry
Geography
Shape Analysis
Biophysics
Biology
Network Analysis
Oncology
…

This leads to the need of:

- *Introducing persistent homology to newcomers*

Combining:

Intuition

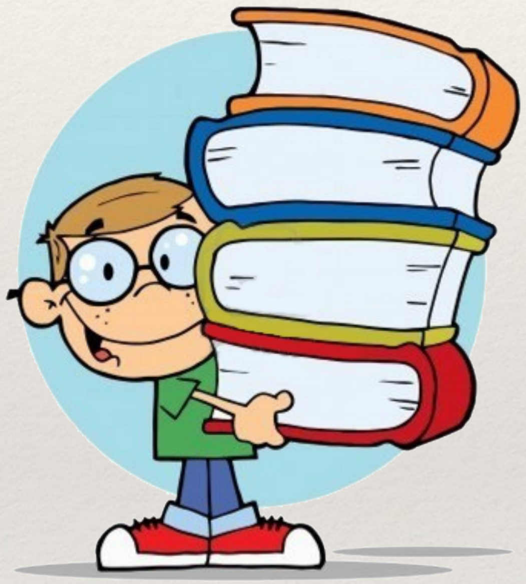


Formalism

Our Contribution

A threefold task:

Interactive website
for beginners



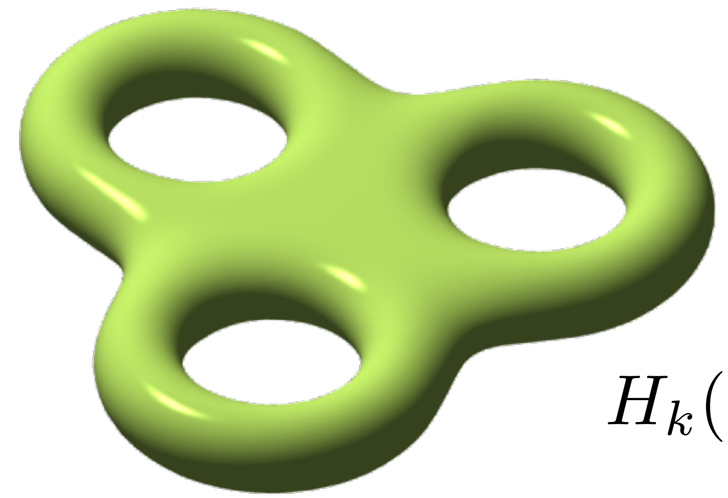
Visualization tool
for curious users

In-depth overview
for interested researchers

Persistent Homology

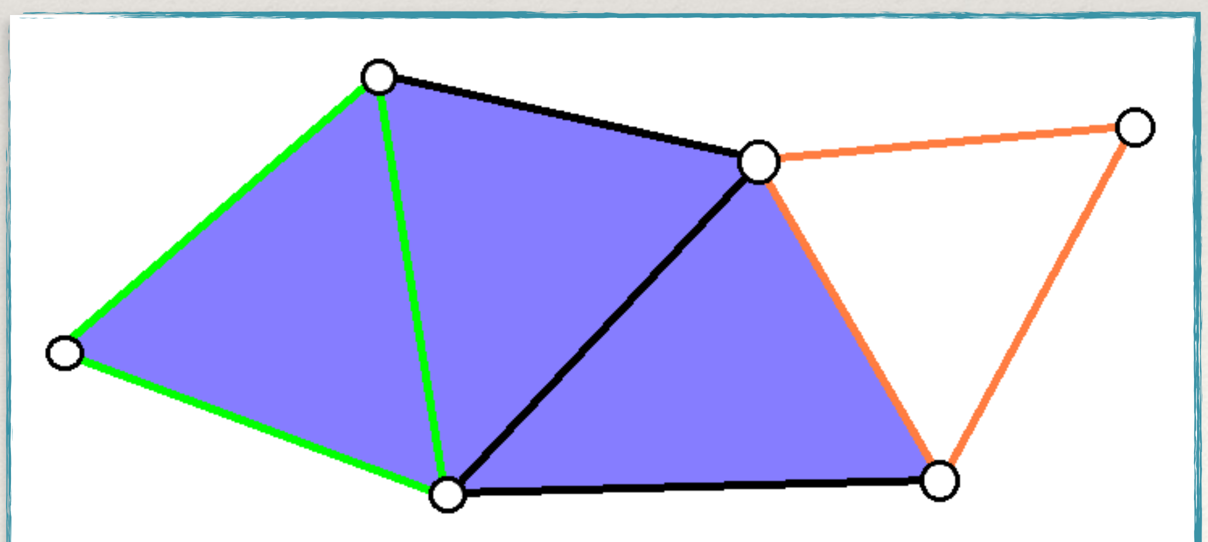
Homology:

A **topological invariant** detecting ***"holes"*** of a shape



$$H_k(\Sigma) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \\ \mathbb{Z}^6 & \text{for } k = 1 \\ \mathbb{Z} & \text{for } k = 2 \end{cases}$$

Given a simplicial complex Σ, the ***k-homology group*** of Σ is defined as



$$H_k(\Sigma) := Z_k / B_k$$

where:

- Z_k is the group of ***k-cycles*** of Σ
- B_k is the group of ***k-boundaries*** of Σ

Persistent Homology

Given a simplicial complex Σ evolving according with a filtration,

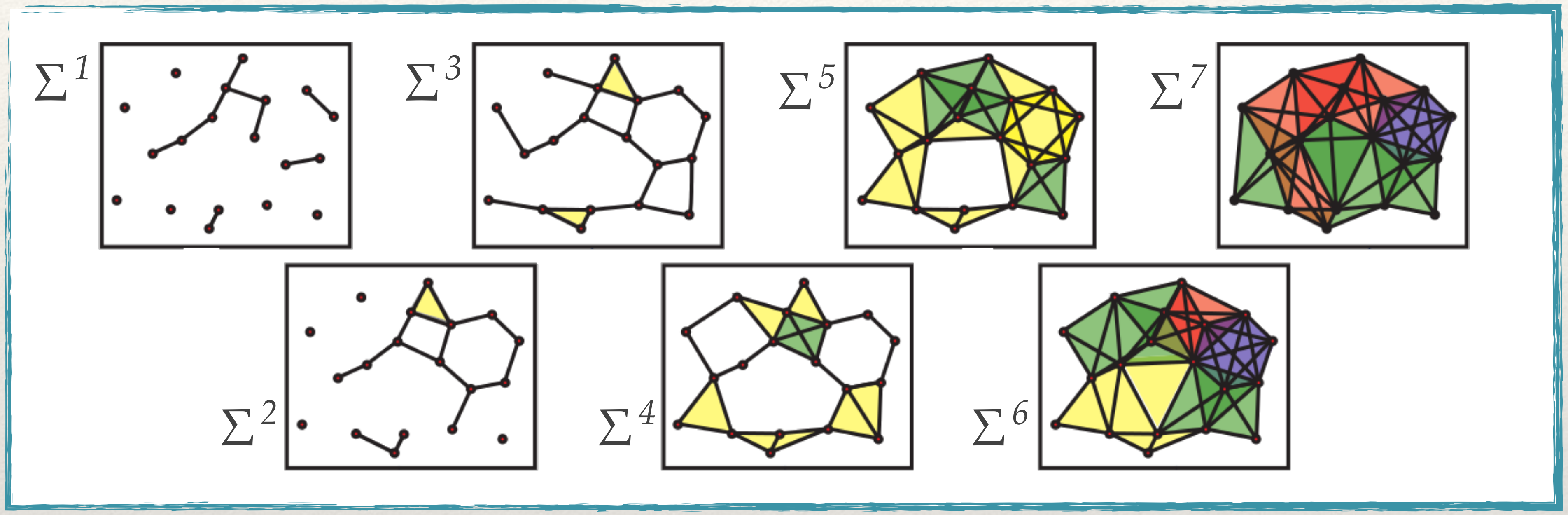


Image from [Ghrist 2008]

The ***(p,q)-persistent k-homology group*** of Σ is defined as

$$H_k^{p,q}(\Sigma) := Im\,(i_k^{p,q})$$

where $i_k^{p,q}$ is the map between $H_k(\Sigma^p)$ and $H_k(\Sigma^q)$ induced by the inclusion of Σ^p in Σ^q

Intuitively:

Persistent homology describes the ***changes in homology*** occurring during the filtration

Defining Persistent Homology

Timeline:

1990

Frosini

Size Functions:

- ***Estimation of natural pseudo-distance*** between shapes endowed with a function *f*
- Tracking of the ***connected components*** of a shape along its evolution induced by *f*

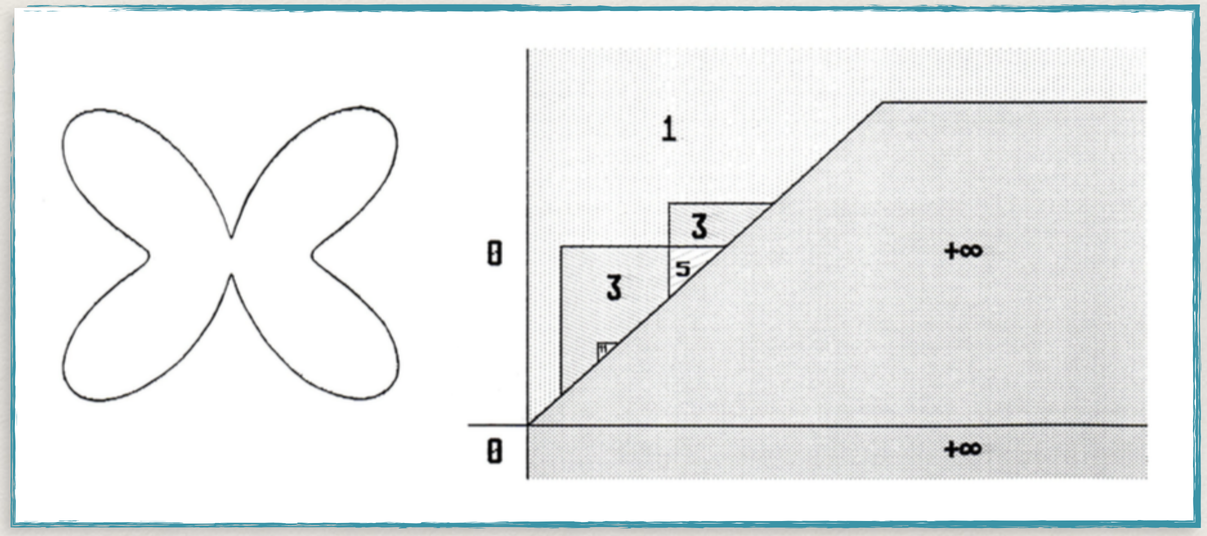


Image from [Frosini 1992]

Actually, this coincides with ***persistent homology in degree 0***

Defining Persistent Homology

Timeline:

1990 — Frosini

1994 — Delfinado, Edelsbrunner

Incremental Algorithm for Betti Numbers:

- Introduction of the notion of *filtration*
- De facto computation of *persistence pairs*

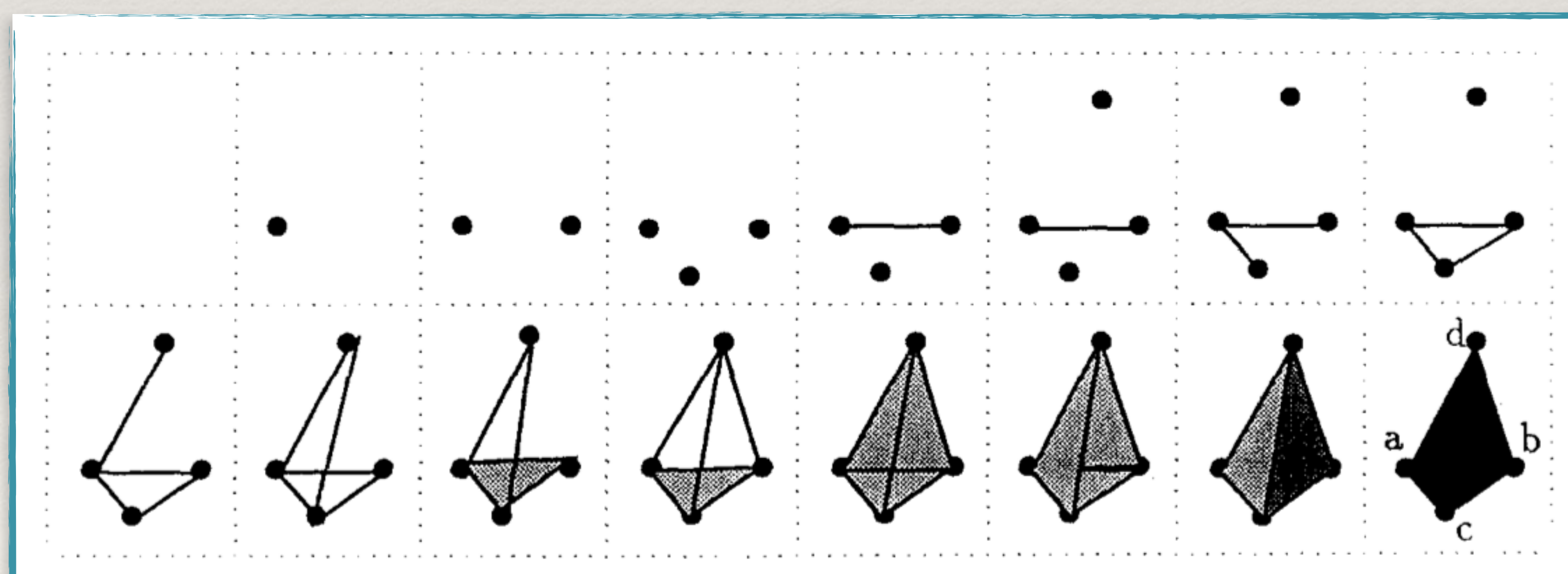


Image from [Delfinado, Edelsbrunner 1995]

Defining Persistent Homology

Timeline:

1990 — Frosini

1994 — Delfinado, Edelsbrunner

1999 — Robins

Homology from Finite Approximations:

- *Extrapolation of the homology* of a metric space from a *finite point-set approximation*
- Introduction of *persistent Betti numbers*

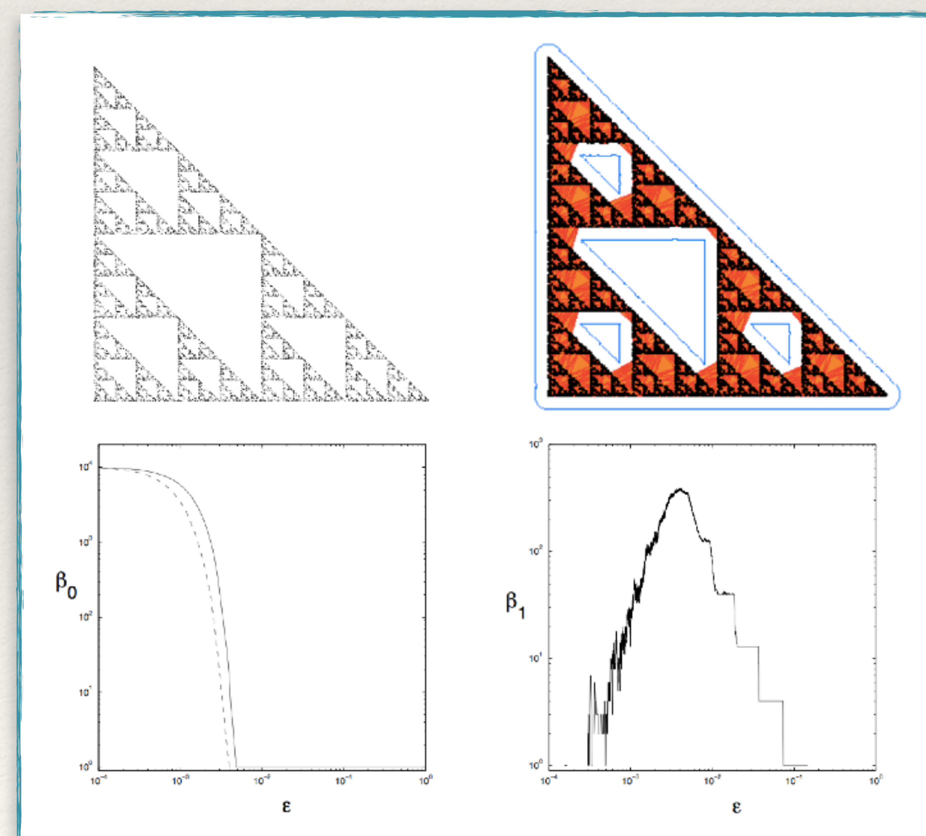


Image from [Robins 1999]

Defining Persistent Homology

Timeline:

1990 — Frosini

1994 — Delfinado, Edelsbrunner

1999 — Robins

2002 — Edelsbrunner, Letscher, Zomorodian

Topological Persistence:

- Introduction and algebraic formulation of the notion of *persistent homology*
- ***Description of an algorithm*** for computing persistent homology

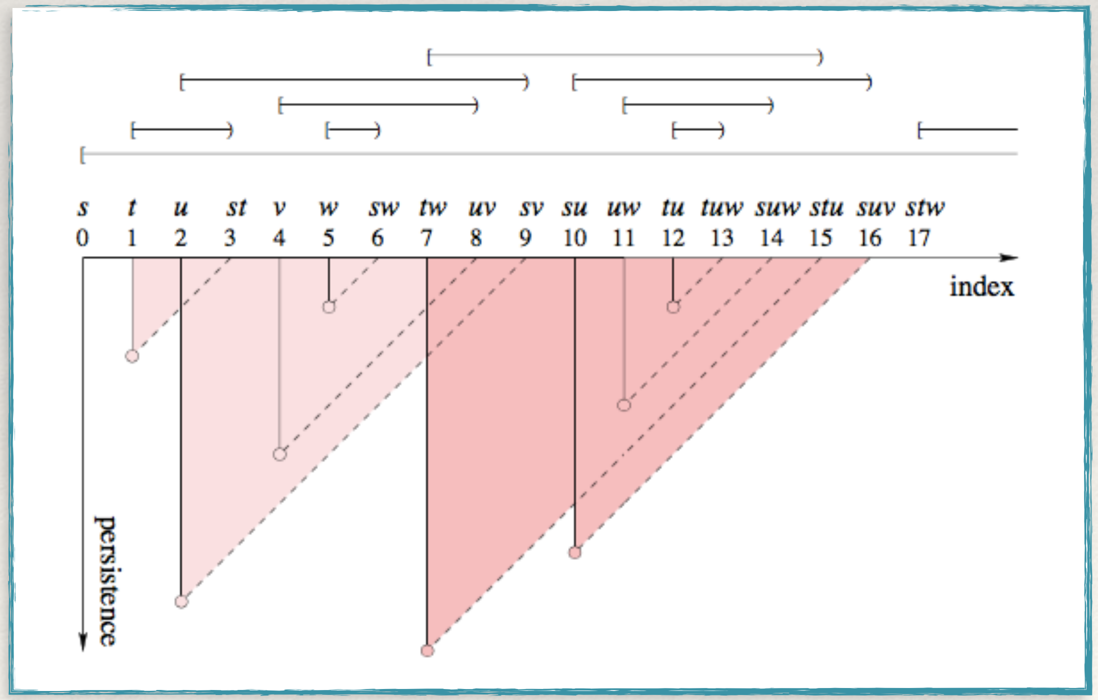
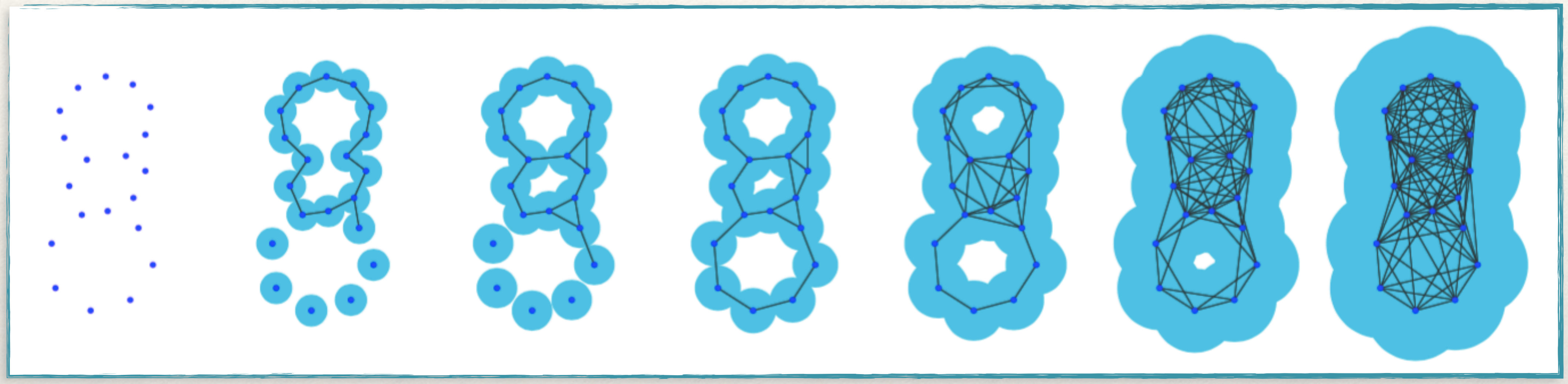


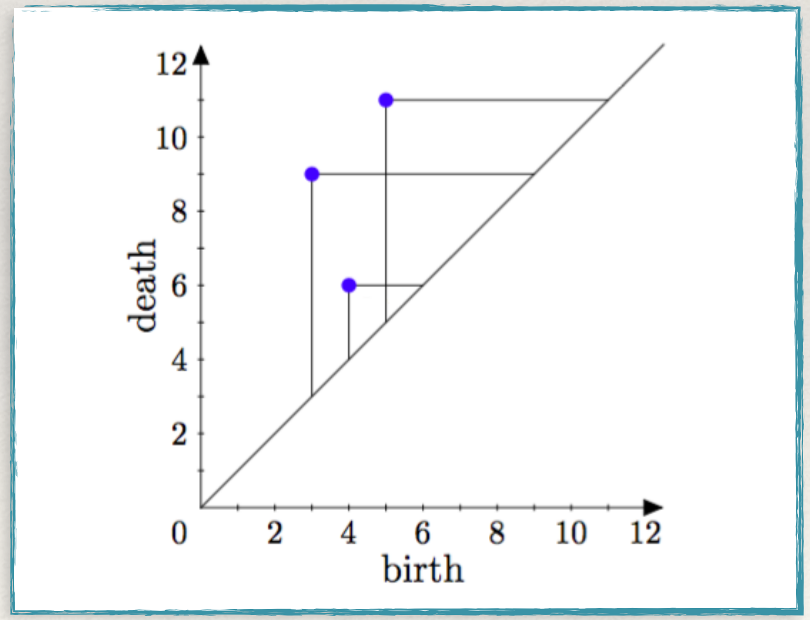
Image from [Edelsbrunner et al. 2002]

Visualizing Persistent Homology

Given a filtered simplicial complex Σ,



Persistent homology of Σ can be visualized through:



- *Persistence diagrams* [Edelsbrunner, Harer 2008]

Images from [Bubenik 2015]

Visualizing Persistent Homology

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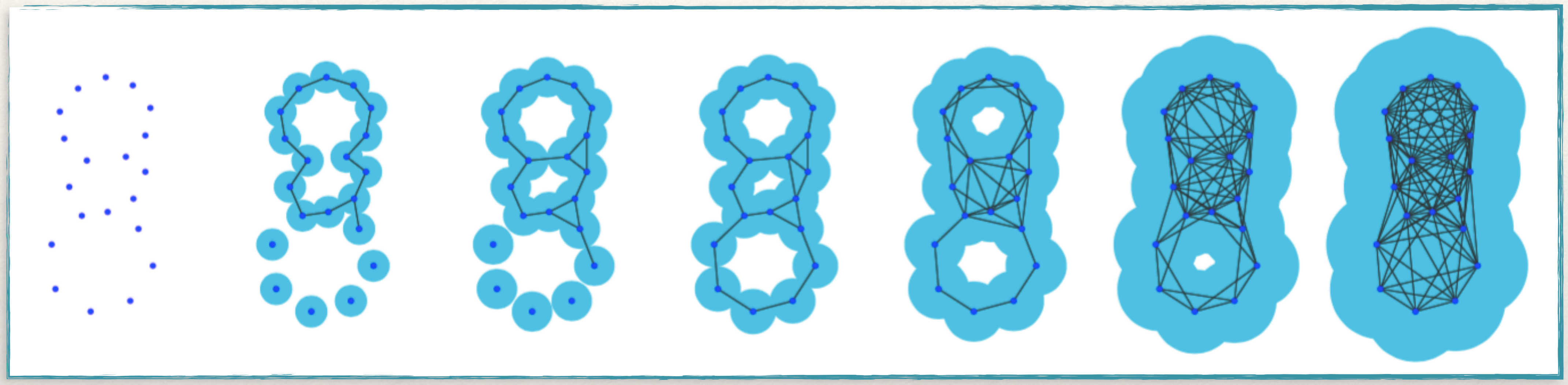

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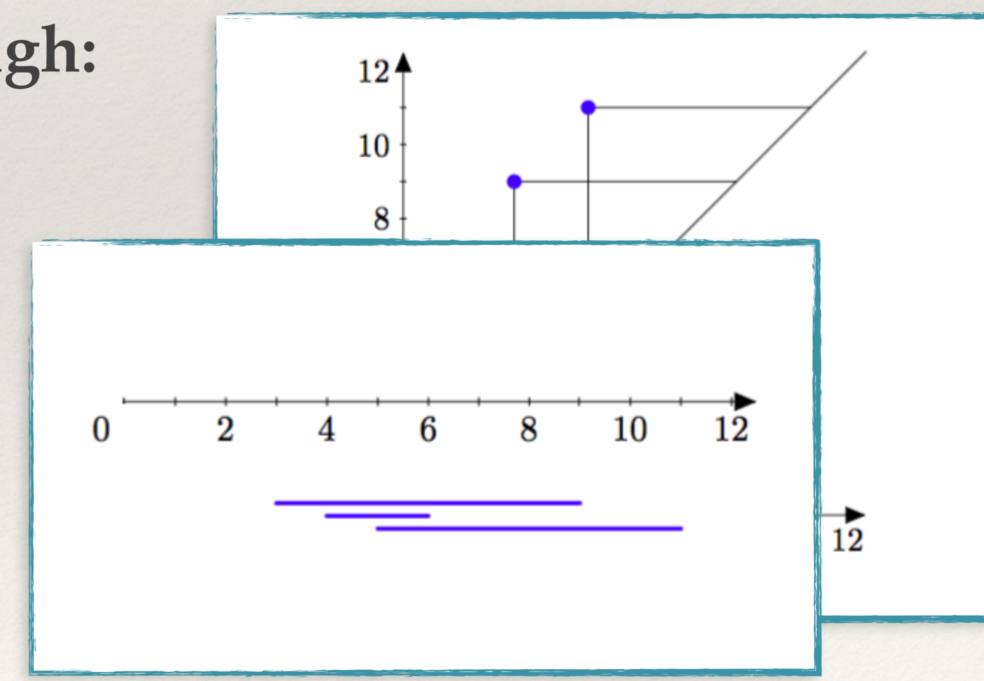
- *Persistence diagrams* [Edelsbrunner, Harer 2008]
- *Barcodes* [Carlsson et al. 2005; Ghrist 2008]



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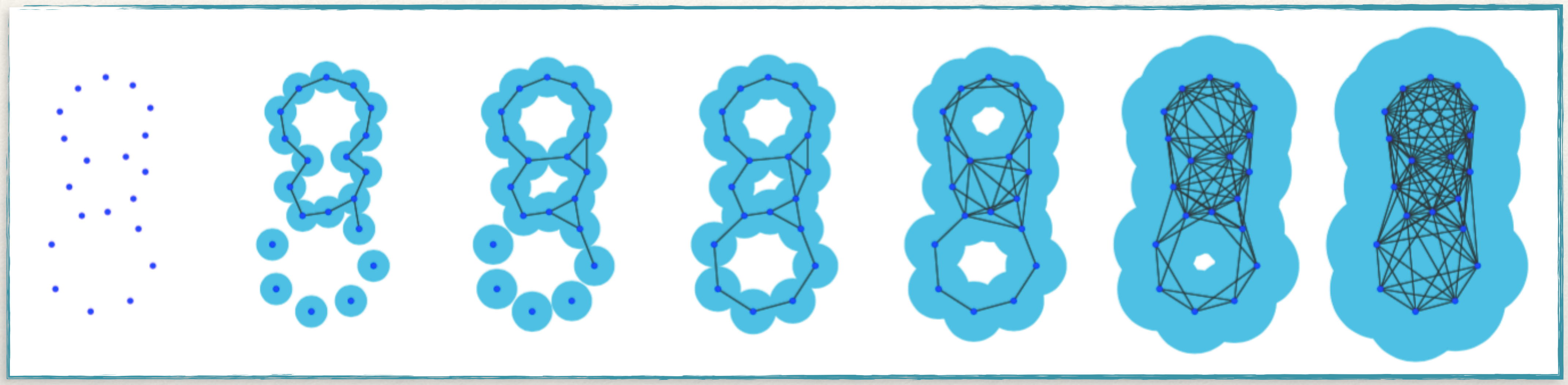
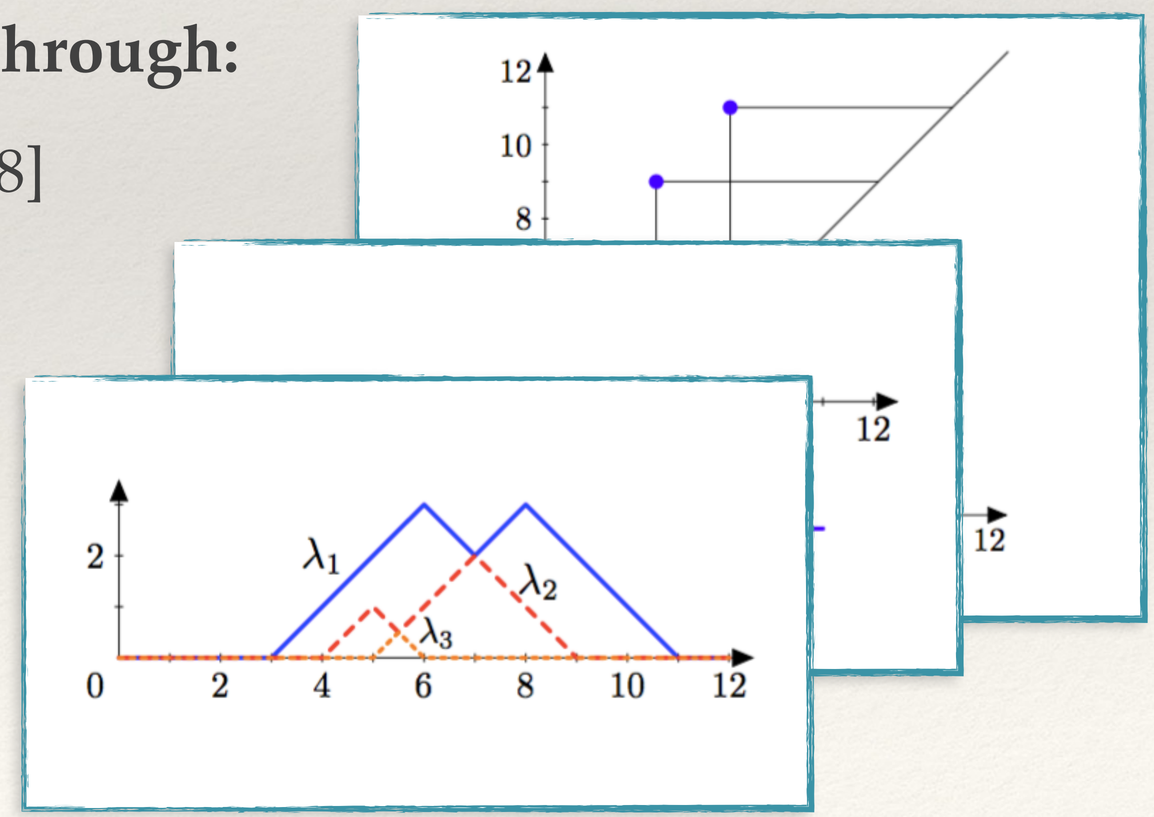


Persistent homology of Σ can be visualized through:

- *Persistence diagrams* [Edelsbrunner, Harer 2008]
- *Barcodes* [Carlsson et al. 2005; Ghrist 2008]
- *Persistence landscapes* [Bubenik 2015]



Images from [Bubenik 2015]

Visualizing Persistent Homology

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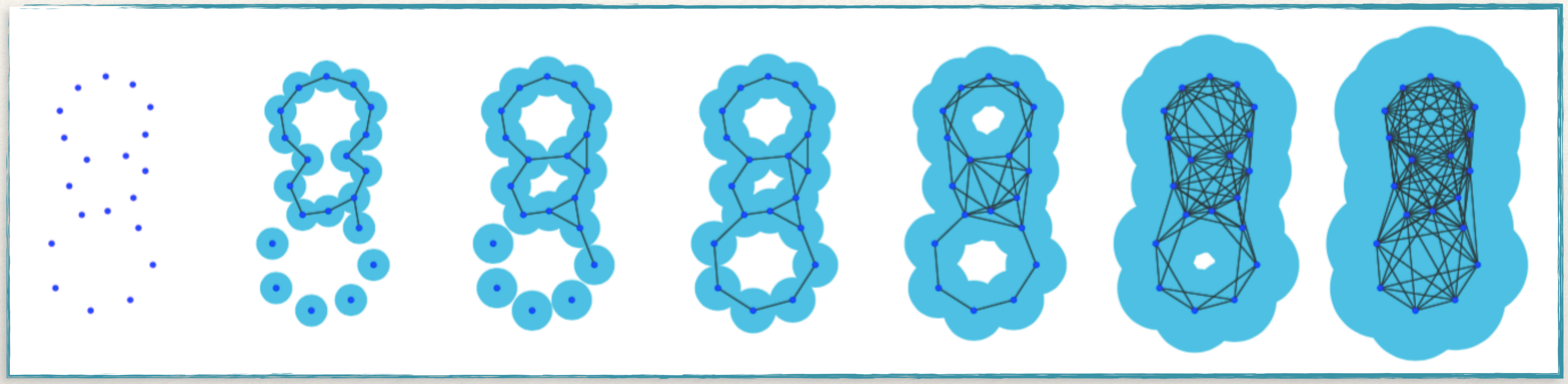
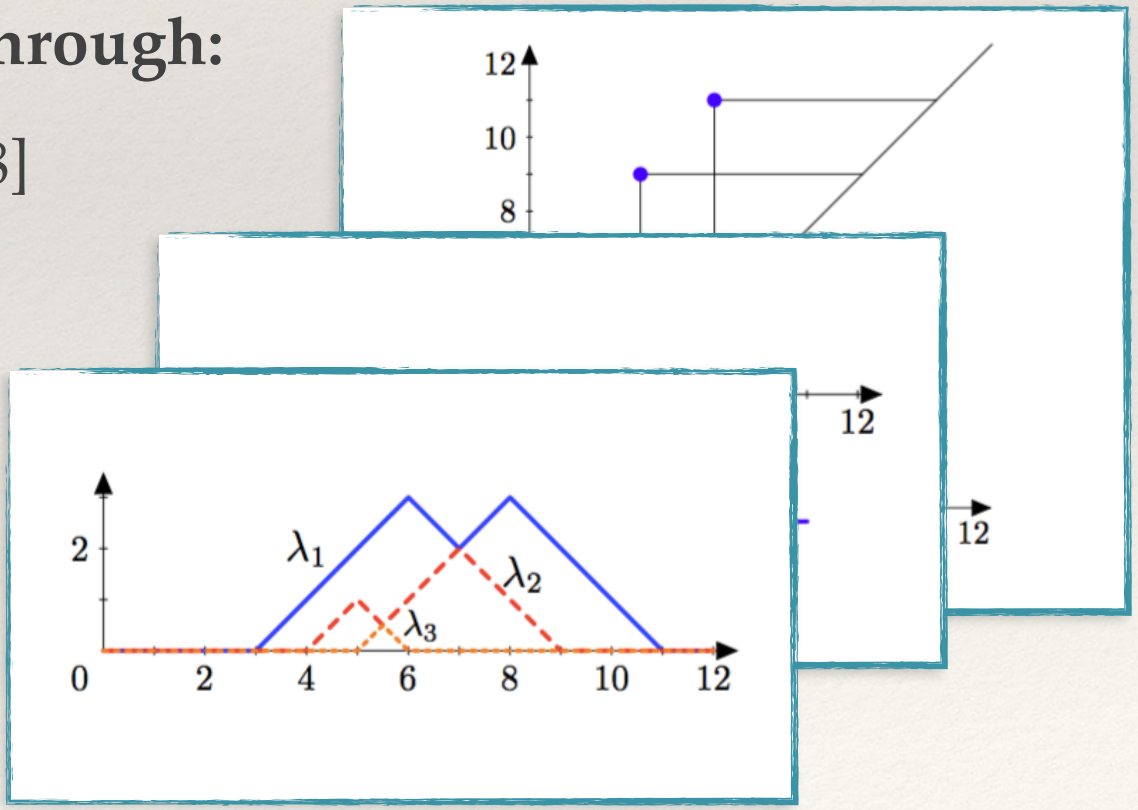


Persistent homology of Σ can be visualized through:

- *Persistence diagrams* [Edelsbrunner, Harer 2008]
- *Barcodes* [Carlsson et al. 2005; Ghrist 2008]
- *Persistence landscapes* [Bubenik 2015]
- *Corner points and lines* [Frosini, Landi 2001]
- *Half-open intervals* [Edelsbrunner et al. 2002]
- *k-triangles* [Edelsbrunner et al. 2002]
- …



Images from [Bubenik 2015]

Computing Persistent Homology

Standard algorithm to compute persistent homology [Zomorodian, Carlsson 2005]:

- Based on a **matrix reduction**
- **Linear complexity** in practical cases
- **Super-cubical complexity** in the worst case

Several different strategies:

Direct optimizations

- *Zigzag persistent homology* [Milosavljević et al. 2005]
- *Computation with a twist* [Chen, Kerber 2011]
- *Dual algorithm* [De Silvia et al. 2011]
- *Output-sensitive algorithm* [Chen, Kerber 2013]
- *Multi-field algorithm* [Boissonnat, Maria 2014]

Coarsening approaches

- *Topological operators and simplifications* [Mrozek, Wanner 2010; Dlotko, Wagner 2014]
- *Morse-based approaches* [Robins et al. 2011; Harker et al. 2014; Fugacci et al. 2014]

Distributed approaches

- *Spectral sequences* [Edelsbrunner, Harer 2008; Lipsky et al. 2011]
- *Multicore coreductions* [Murty et al. 2013]
- *Multicore homology* [Lewis, Zomorodian 2014]
- *Persistent homology in chunks* [Bauer et al. 2014a]
- *Distributed persistent computation* [Bauer et al. 2014b]

Annotation-based methods

- *Compressed annotation matrix* [Boissonnat et al. 2013]
- *Persistence for simplicial maps* [Dey et al. 2014]

Interactive User-Guide

We propose a web-based user-guide on persistent homology for beginners and researchers coming from other fields

Main Contributions:

- Intuitive and self-contained ***introduction to persistent homology***
- Step-by-step ***description of the standard algorithm*** for computing persistent homology
- ***Overview of the state of the art*** in persistent homology

Intuitiveness & Interactivity

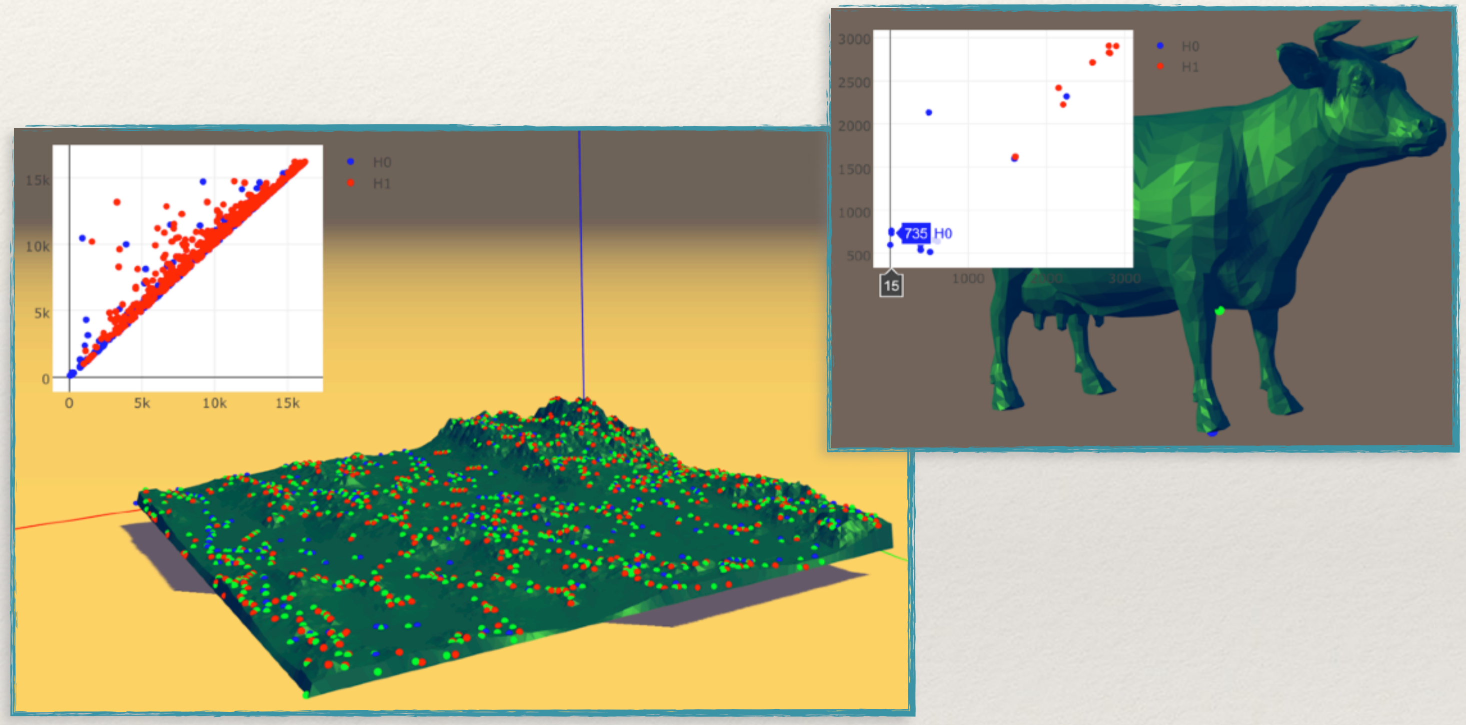
- *Accessible language and elementary definitions*
- *Step-by-step descriptions of notions and algorithms*
- *Different colors for different tasks*
- *Interactive examples*
- *Focus on Z_2 coefficients and standard algorithm*

Formalism & Completeness

- *Self-contained and theoretically consistent*
- *Pseudo-code description of the algorithms*
- *Classification of the state of the art*
- *Insights on some relevant theoretical aspects*
- *Links to relevant contents and cited works*

Web-GL Interface

We develop a visualization tool for studying persistence pairs on a triangulated surface



Two Packages:

1. From *a filtered simplicial complex* Σ to *its persistence pairs*
2. From *the persistence pairs of* Σ to *their visualization in an interactive interface*

Web-GL Interface

1. **Computing Persistent Homology**

- **Accepted input:**
  - ***Σ, triangulated surface*** (supported formats: .ply, .off)
  - $f : \Sigma_0 \longrightarrow \mathbb{R}$, ***filtering function defined on the vertices of Σ***
- **Computation of the persistence pairs:**
  - based on the ***standard algorithm implemented in*** **PHAT Library** [Bauer et al. 2013]

2. **Visualizing Persistence Pairs**

- **Accepted input:**
  - ***Σ, triangulated surface*** (supported formats: .ply)
  - ***PP, list of the persistence pairs of Σ*** (supported formats: .json)
- **Visualization of the persistence pairs of Σ through:**
  - ***3D scene,*** implemented using the **Threejs Library**, *a Javascript library based on Web-GL*
  - ***scatter plot,*** implemented using the **Plotly Library**

Current and Future Developments

In Summary:

- A new approach for ***spreading persistent homology as a practical tool*** has been proposed
- It consists of:
  - ***Interactive web-based user-guide*** introducing persistent homology
  - ***Web-GL interface*** for analyzing persistence pairs on a triangulated surface
  - ***In-depth overview*** on the evolution of persistent homology and the state-of-the-art methods

What's Next?

- We are planning to ***expand the user-guide and the visualization tool*** including:
  - ***Morse theory***
  - ***Forman's discrete Morse theory***
  - ***Reeb graphs***
- **Long-term goal:** ***a shared framework where researchers can participate in building user-friendly guides***

Thank you

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Joint work with:

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