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TOPOLOGICALLY-CONSISTENT SIMPLIFICATION OF DISCRETE MORSE COMPLEXES

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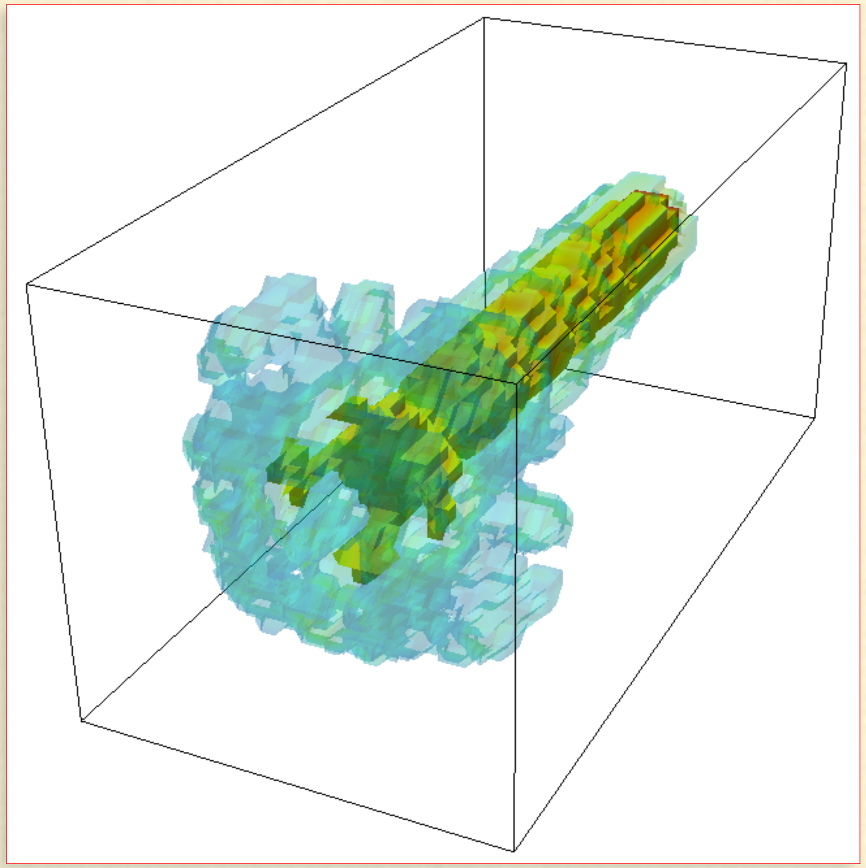
MOTIVATION

Morse Theory is a fundamental tool for studying the morphology of a scalar field defined on a shape



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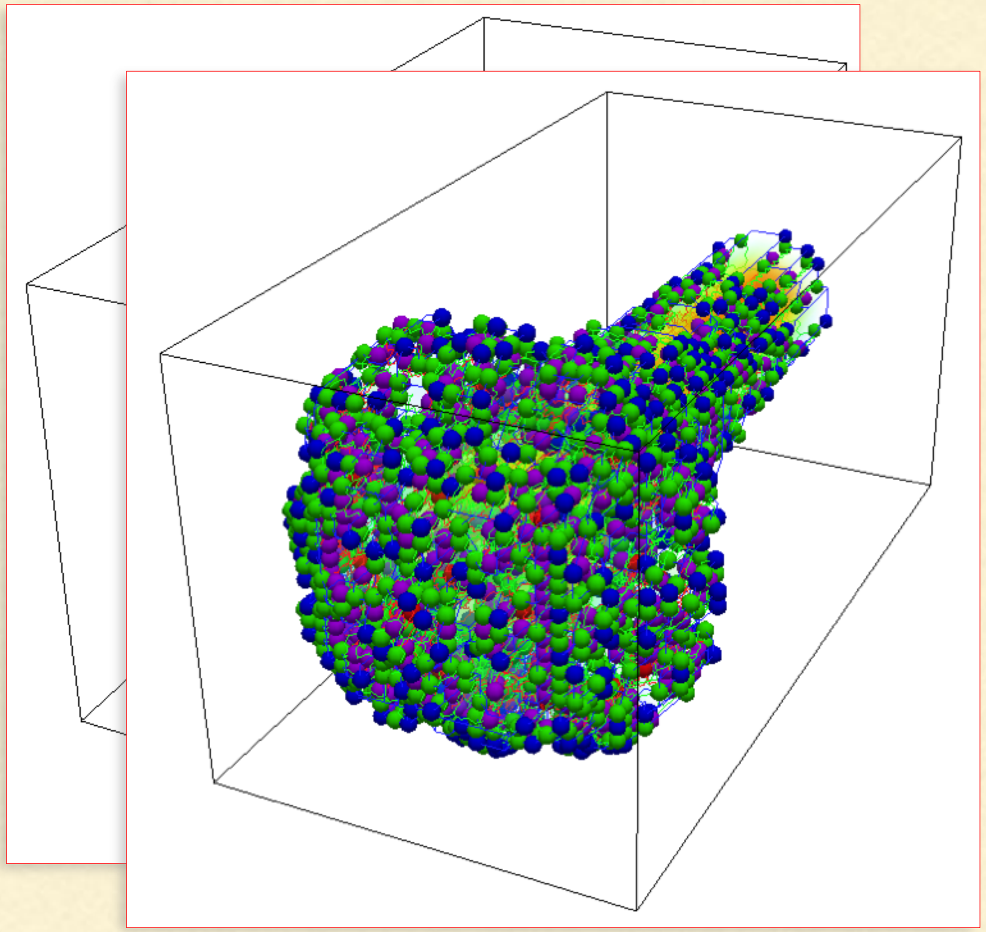

Working with real data,

- size of the morphological segmentation
  - presence of noise

requires a **morphological simplification** of the dataset

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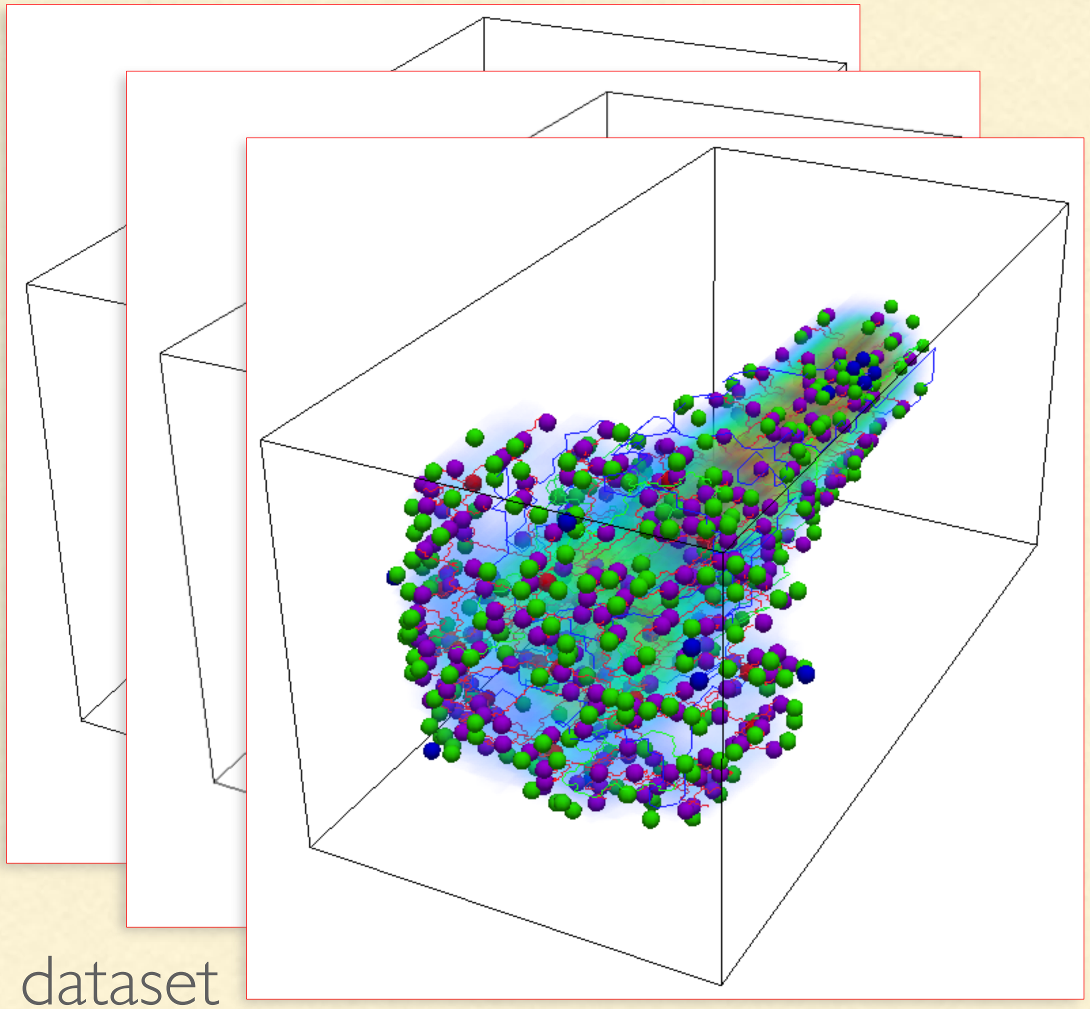

Working with real data,

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MOTIVATION

Two issues affect morphological simplification:

- Lack of a **data structure** for Morse complexes combining
  - compactness in storage cost
  - efficiency for interactive modifications
- Topological **inconsistencies** between two different simplification methods

Our contribution:

- A new **compact** and **efficient** data structure
- A new **simplification algorithm** ensuring **topological consistency**

OUTLINE

- **Background Notions**
  - Discrete Morse Theory
  - Morse Complexes
- **Representing Morse Complexes**
  - Gradient-based and Graph-based Representations
  - Discrete Morse Incidence Graph (DMIG)
- **Simplifying Morse Complexes**
  - Topological Inconsistencies during the Simplification
  - Shared *V*-path Disambiguation
- **Simplification Algorithm**
  - Topologically-Consistent Simplification Algorithm
  - Experimental Results

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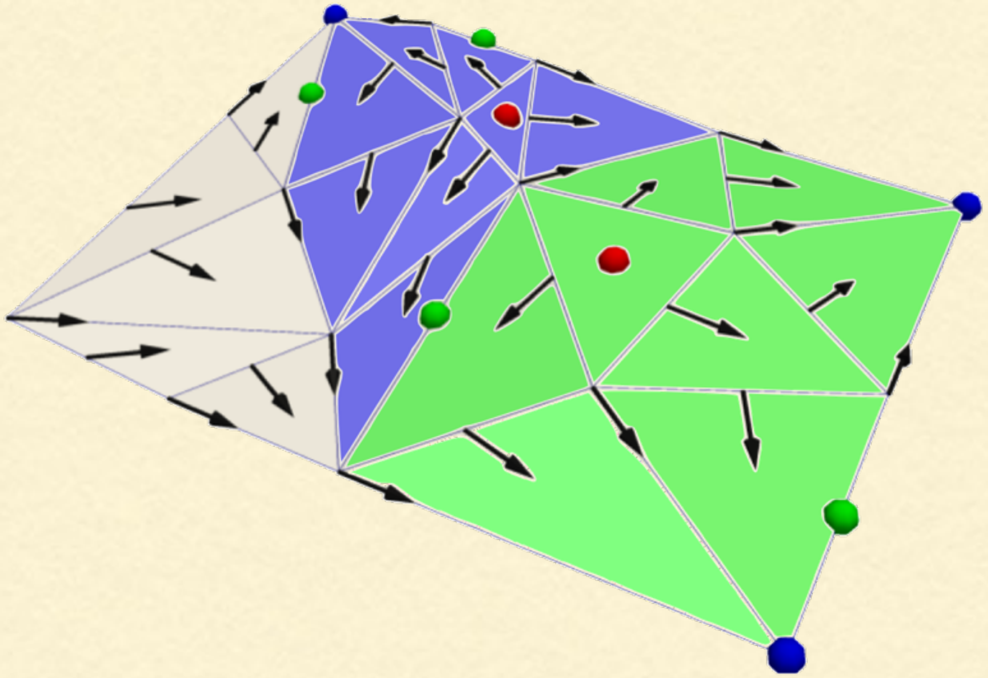
DISCRETE MORSE THEORY [FORMAN 1998]

Discrete Morse theory is a combinatorial counterpart of Morse theory defined for cell complexes

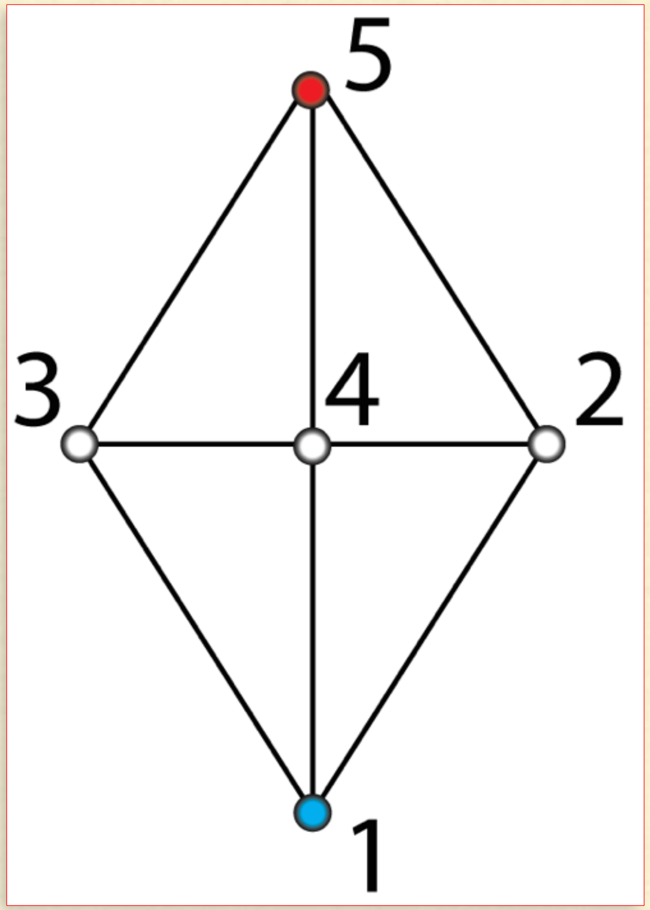


Through the analysis of the critical cells of a function defined on a discretized shape,

- gives a compact homology-equivalent model for a shape
- is a tool for computing segmentations of shapes

DISCRETE MORSE THEORY

Let Σ be simplicial complex endowed with a function f defined on its vertices



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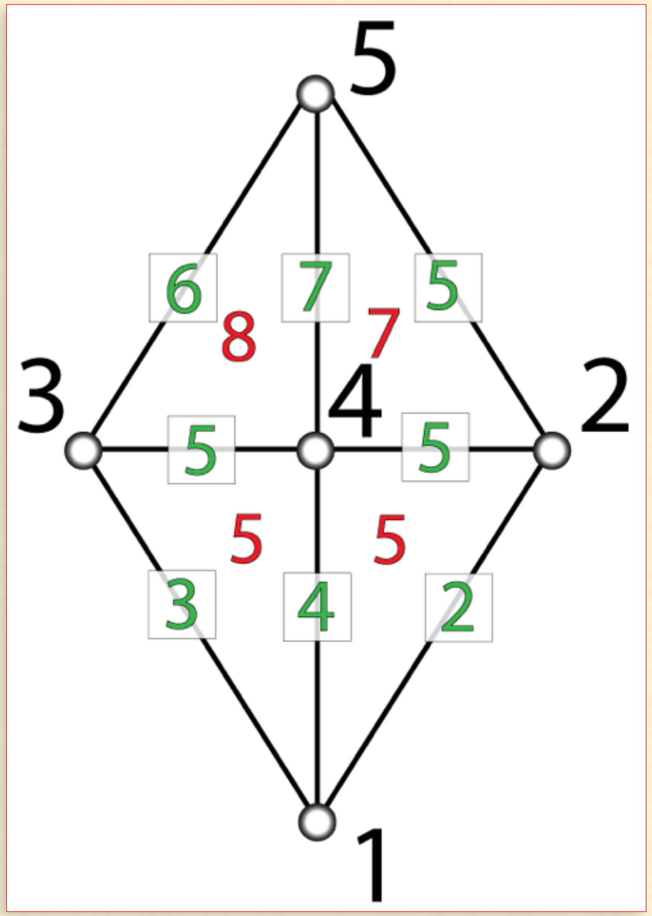

Discrete Morse theory allows to

- extend *f* to all simplices

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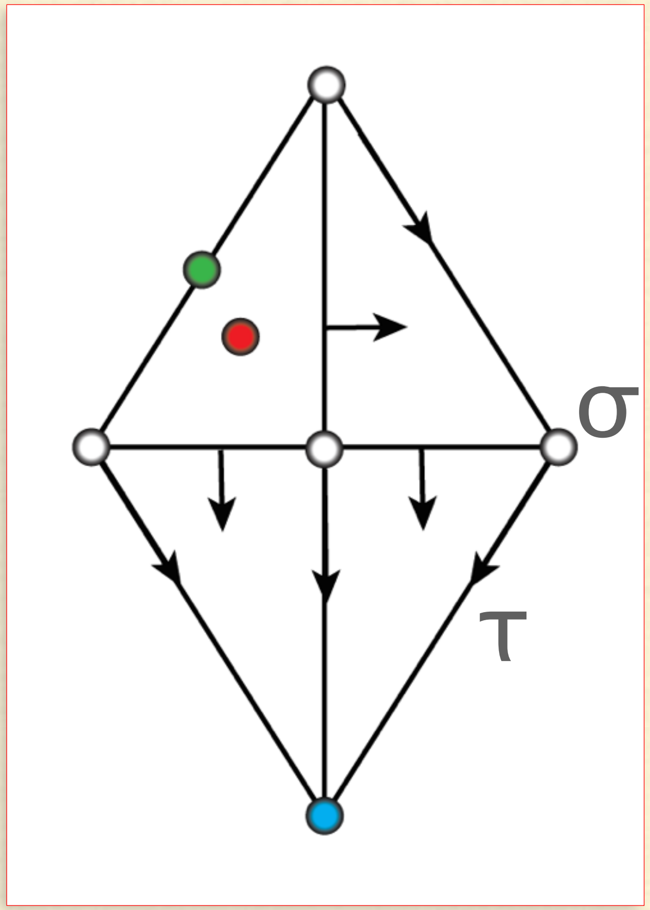
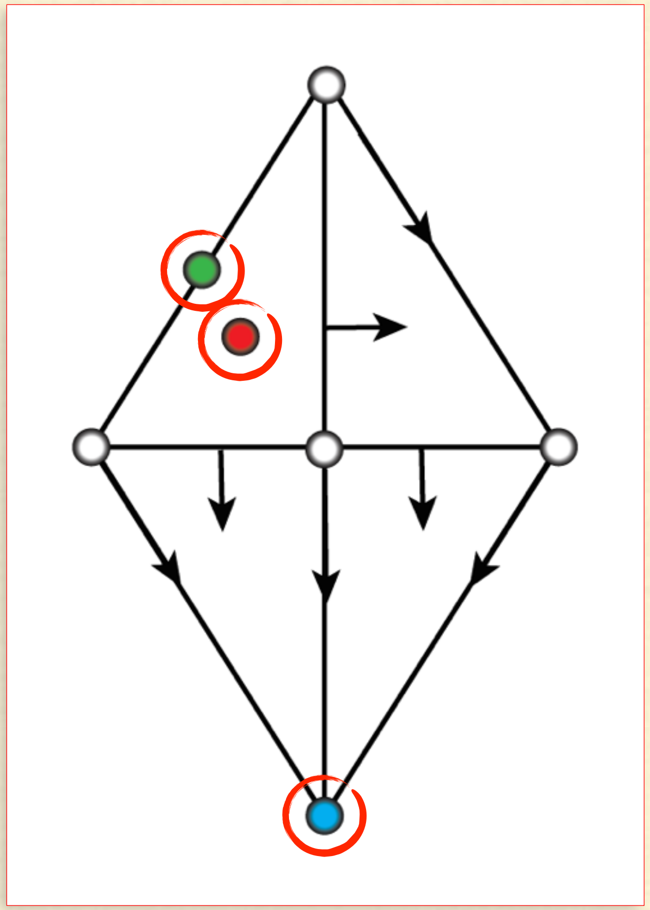


Discrete Morse theory allows to

- extend *f* to all simplices
- build a **gradient vector field *V*** on Σ
  - each pair (**σ**,**τ**) in *V* is an arrow from a *k*-simplex **σ** to a *(k+1)*-simplex **τ**

DISCRETE MORSE THEORY

Unpaired simplices of dimension k are denoted as **critical** simplices of index k



DISCRETE MORSE THEORY

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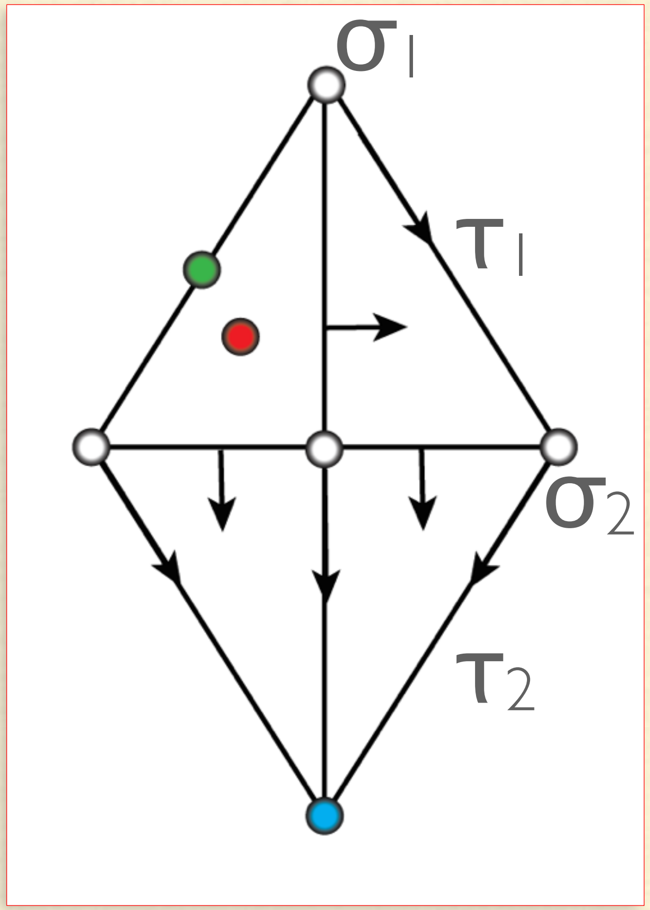


A ***V*-path** is a collection of pairs of *V*

$(\sigma_1,\tau_1), (\sigma_2,\tau_2), \ldots, (\sigma_{r-1},\tau_{r-1}), (\sigma_r,\tau_r)$

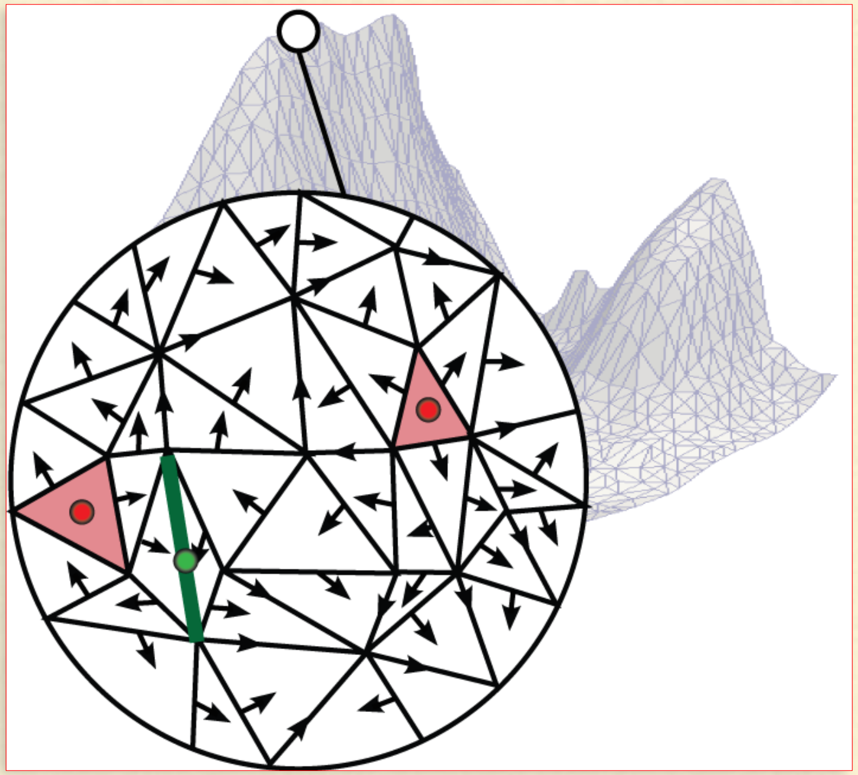
such that

- σ_{i+1} is a *k*-simplex face of the *(k+1)*-simplex τ_i
- σ_{i+1} is different from σ_i

Each gradient vector field *V* built using discrete Morse theory is free of closed *V*-paths

MORSE COMPLEXES

Let Σ be a simplicial complex of dimension d



Navigating the *V*-paths, one can retrieve:

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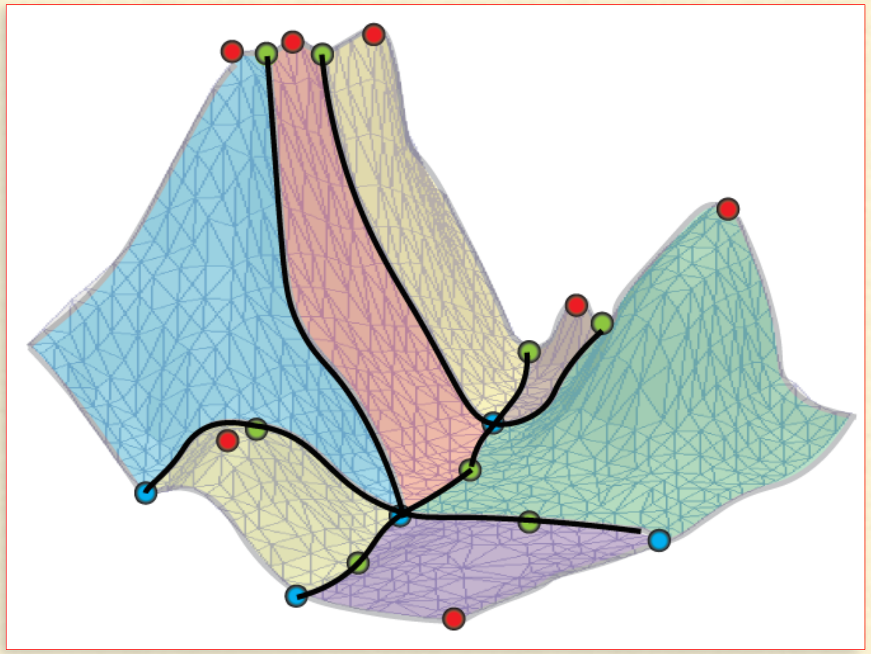

Navigating the *V*-paths, one can retrieve:

- **Descending** Morse complex Γ_D
  - generated by collection of the d-cells representing the regions of influence of the maxima of f : ***k*-cells of $\Gamma_D \longleftrightarrow$ critical simplices of index *k***

MORSE COMPLEXES

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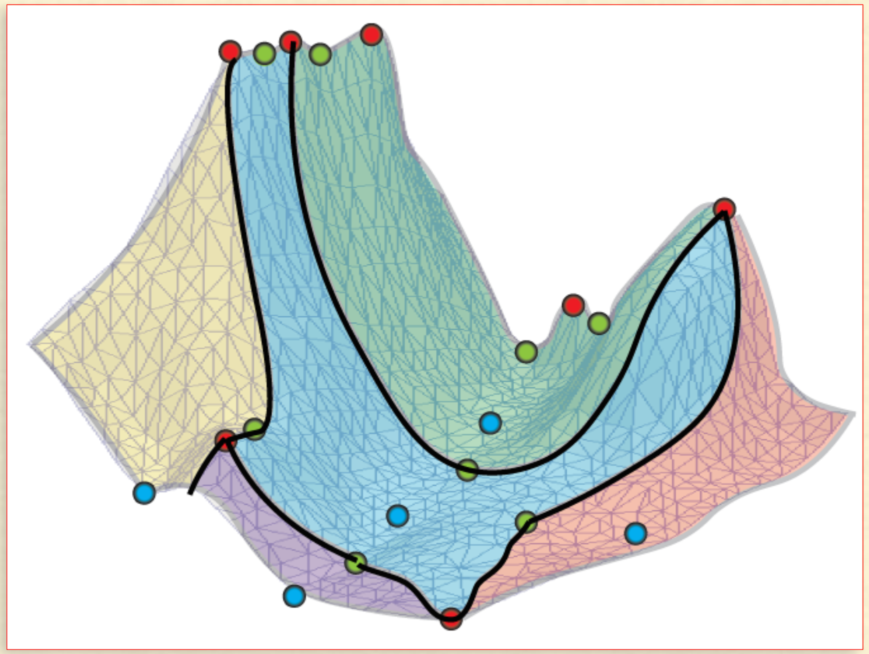
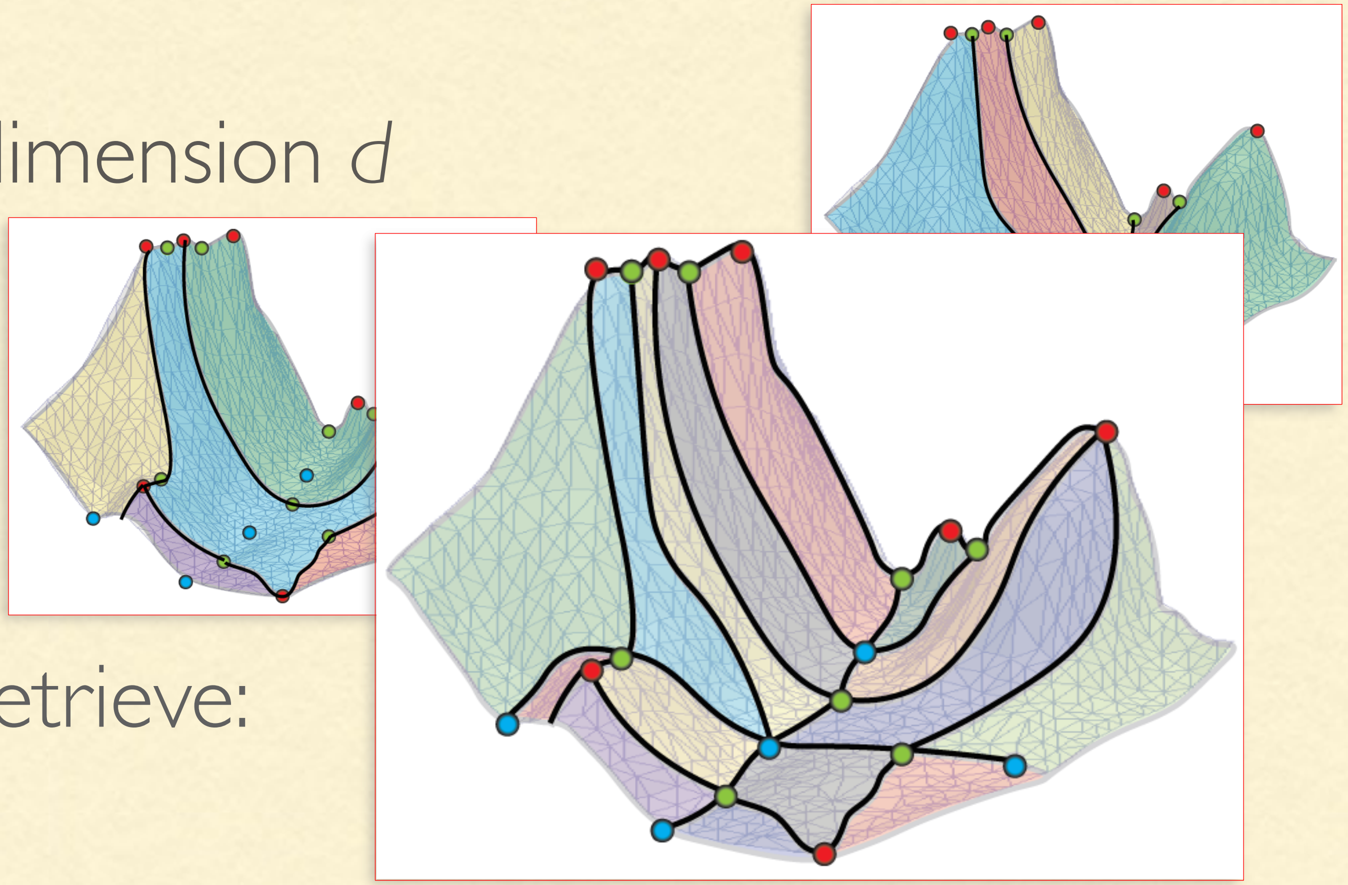


Navigating the *V*-paths, one can retrieve:

- **Ascending** Morse complex Γ_A
  - generated by collection of the *d*-cells representing the regions of influence of the minima of *f* : **(*d*-*k*)-cells of $\Gamma_A \longleftrightarrow$ critical simplices of index *k***

MORSE COMPLEXES

Let Σ be a simplicial complex of dimension d



Navigating the *V*-paths, one can retrieve:

- **Morse-Smale** complex $\mathbf{\Gamma}_{MS}$
  - generated by the connected components of the intersection of the cells of the descending and ascending Morse complexes

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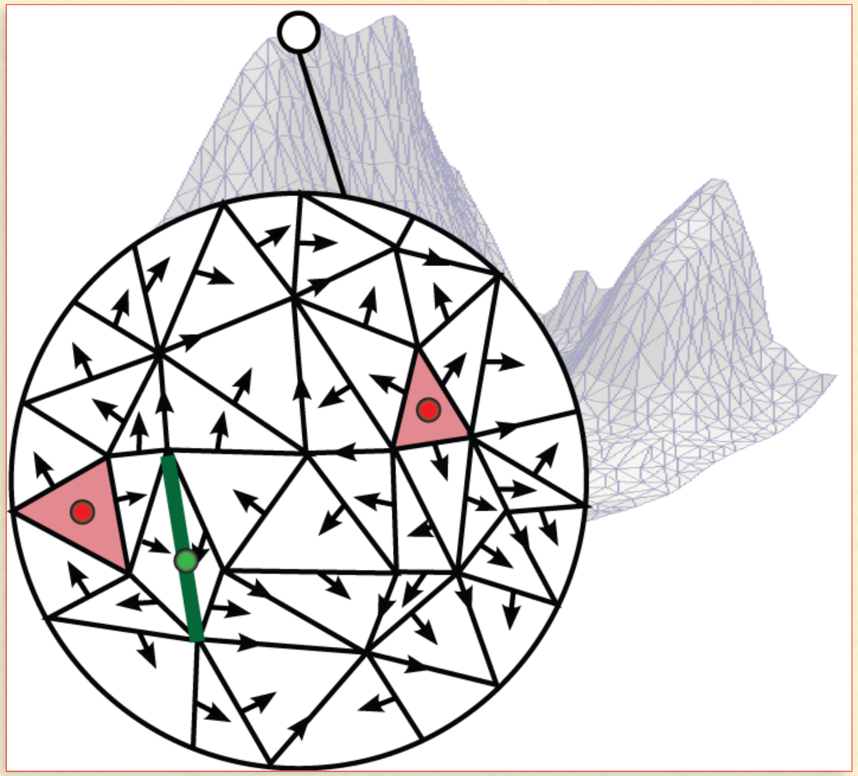
REPRESENTING MORSE COMPLEXES

Two kinds of representation are used for Morse complexes:

- Implicit representation
  - **Gradient-based**
- Explicit representation
  - **Graph-based**

Both the representations require a data structure for encoding the underlying simplicial complex Σ

REPRESENTING MORSE COMPLEXES: GRADIENT-BASED REPRESENTATION



Gradient-based representation encodes the arrows defining the gradient vector field *V*

Gradient *V* can be encoded

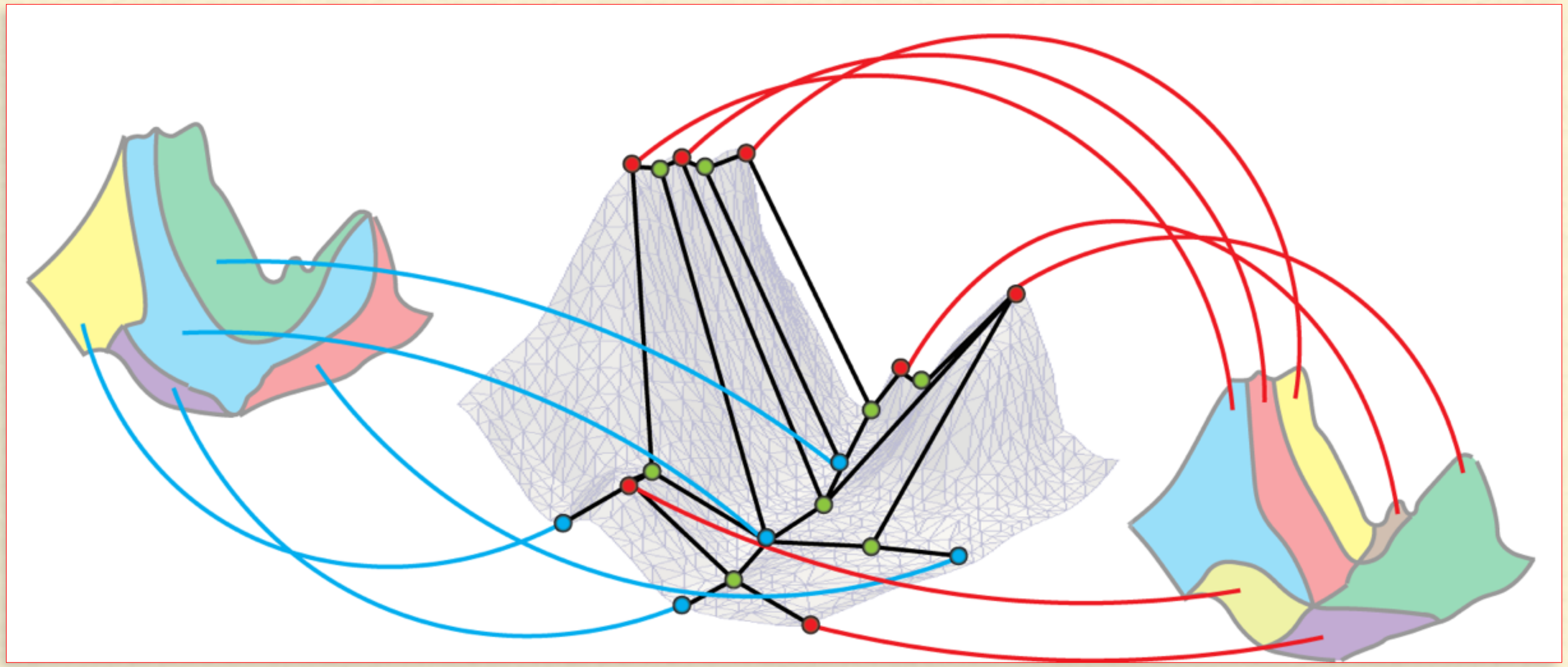
using an Incidence Graph data structure for Σ

- through a **Boolean value** for each incidence relation between two simplices

or, **more compactly,** using the IA* data structure for Σ

- through a **bitvector** for each top simplex of Σ [Weiss et al. 2013]

REPRESENTING MORSE COMPLEXES: GRAPH-BASED REPRESENTATION



Graph-based representation consists of

- **Morse Incidence Graph (MIG)**: a weighted graph whose
  - nodes ⟷ Morse cells
  - arcs encodes incidence relations between two Morse cells
- For each node of the MIG, the **entire geometrical embedding** of the corresponding Morse cell

REPRESENTING MORSE COMPLEXES

Gradient-based Representation

+ compact data structure
− inefficient in updates

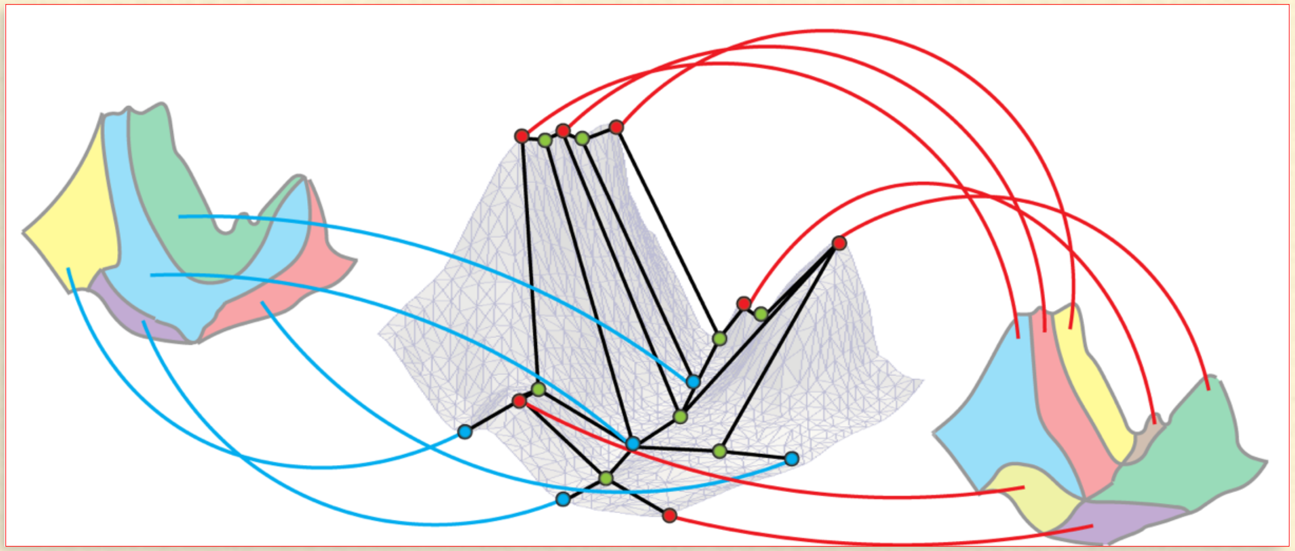
Graph-based Representation

+ generally faster for updates
− high storage cost

We propose a new data structure for Morse complexes coupling **compactness** and **efficiency**

REPRESENTING MORSE COMPLEXES: DMIG

Combining gradient-based and graph-based representation,
we have defined the **Discrete Morse Incidence Graph (DMIG)**

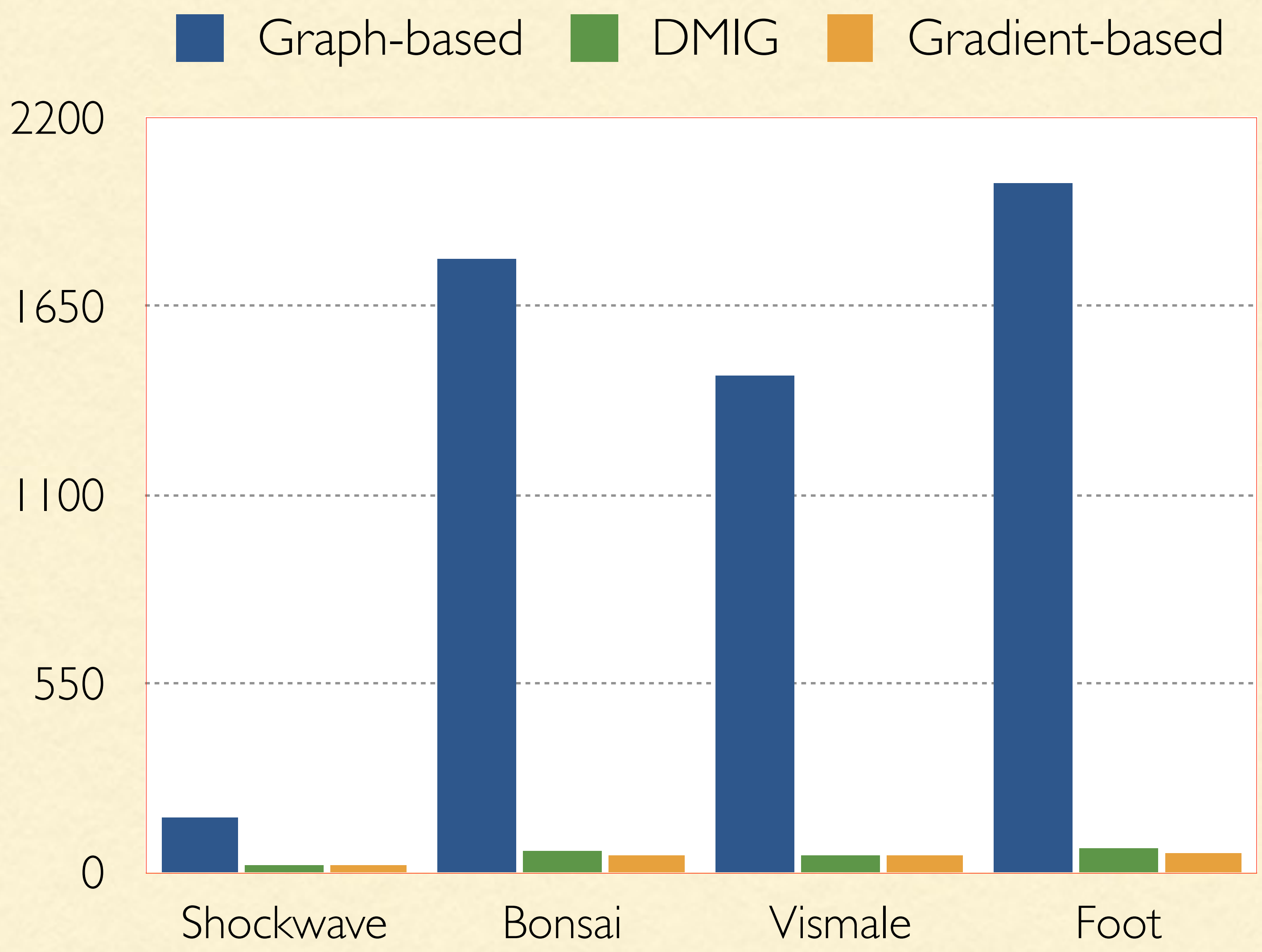


DMIG consists of

- Compact gradient encoding
- Morse Incidence Graph (MIG)
- For each node of the MIG, the critical simplex of the corresponding Morse cell
  - a single simplex instead of the entire geometrical embedding

REPRESENTING MORSE COMPLEXES: DMIG

Storage cost of the DMIG with respect to Graph-based and Gradient-based representation



DMIG results to be

- 7 to 30 more compact than the graph-based representation
- always comparable with the gradient-based representation

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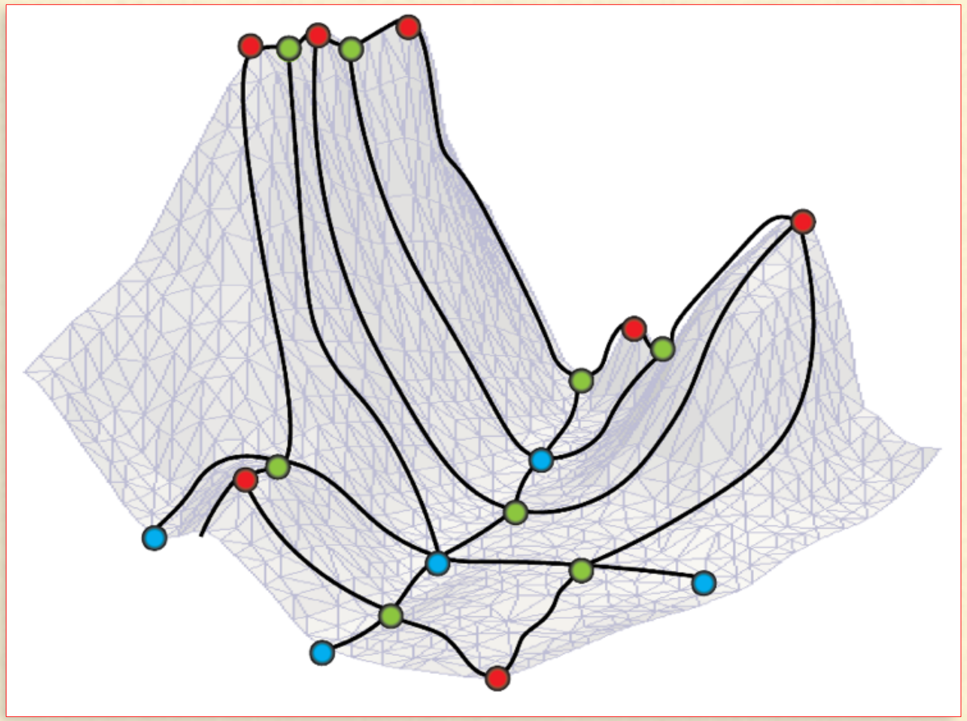
SIMPLIFYING MORSE COMPLEXES

Topology-based simplification of scalar fields is a powerful tool for

- Removing insignificant features
- Preserving relevant parts of the data



Simplification algorithms perform elementary simplification operators organized in a sequence with respect to a chosen priority measure

- Persistence [Edelsbrunner et al. 2002]
- Separatrix persistence [Weinkauf et al. 2009]
- Topological saliency [Doraiswamy et al. 2013]

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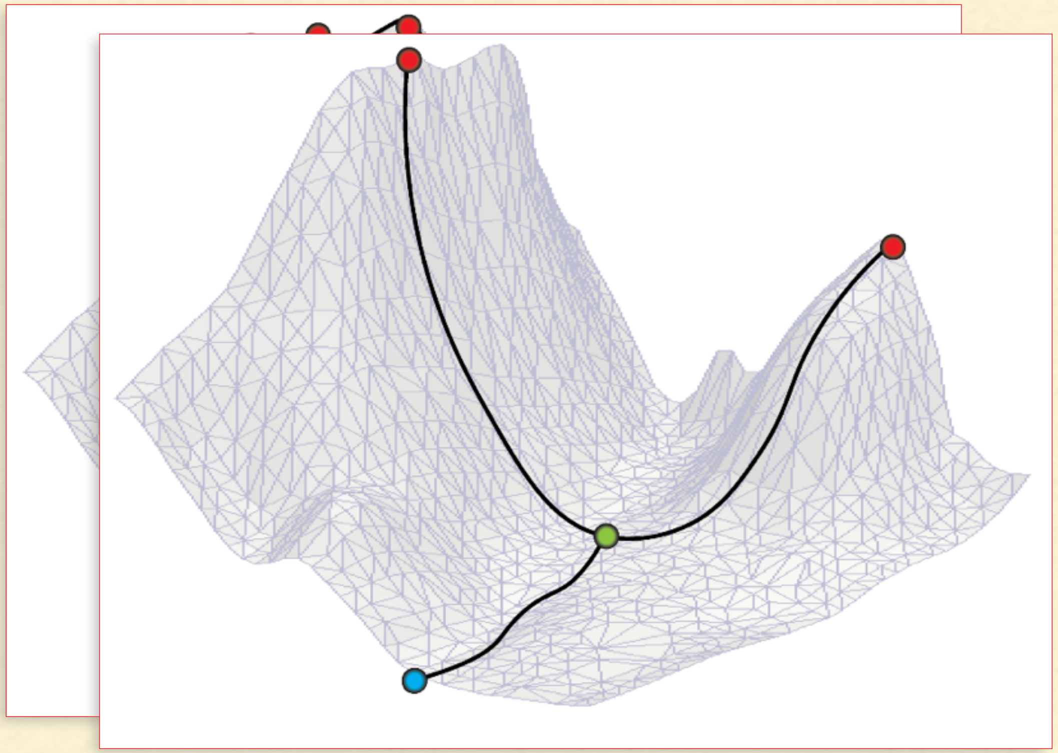

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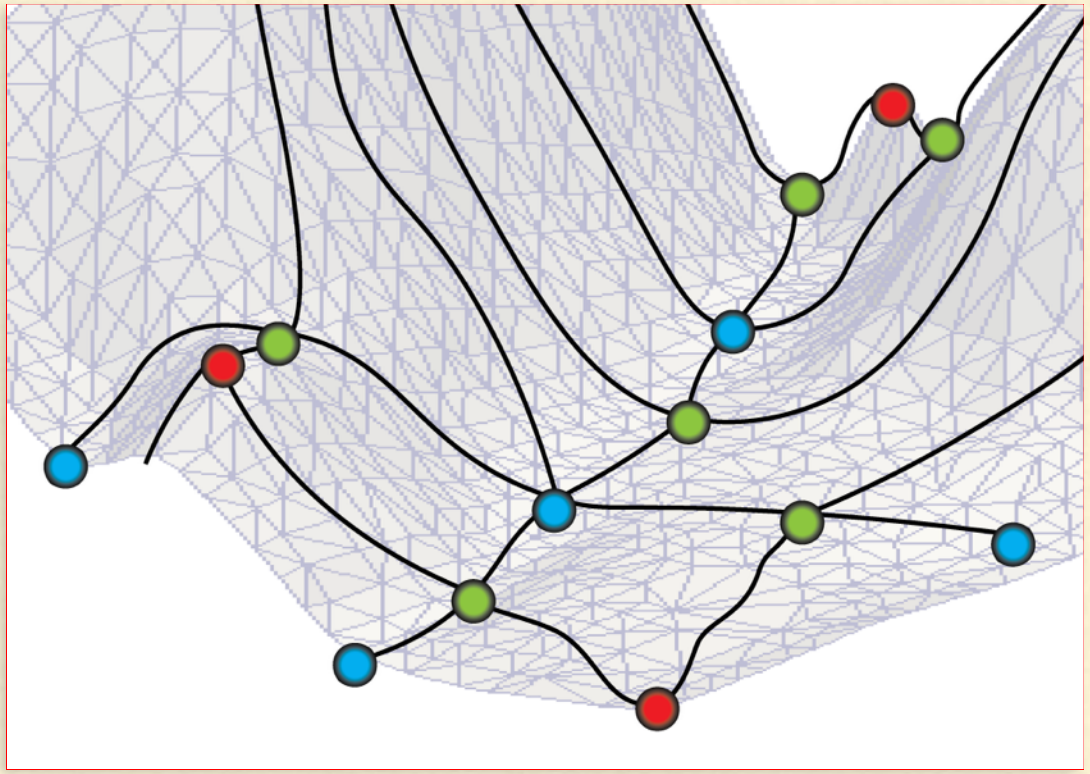
SIMPLIFYING MORSE COMPLEXES: CANCELLATION OPERATOR

The most common simplification operator is called **cancellation** [Forman, 1998]



k-cancellation(σ,τ) removes a pair of critical simplices of index *k* and *k* + *1* respectively under the assumption that

- σ and τ are connected by a unique *V*-path

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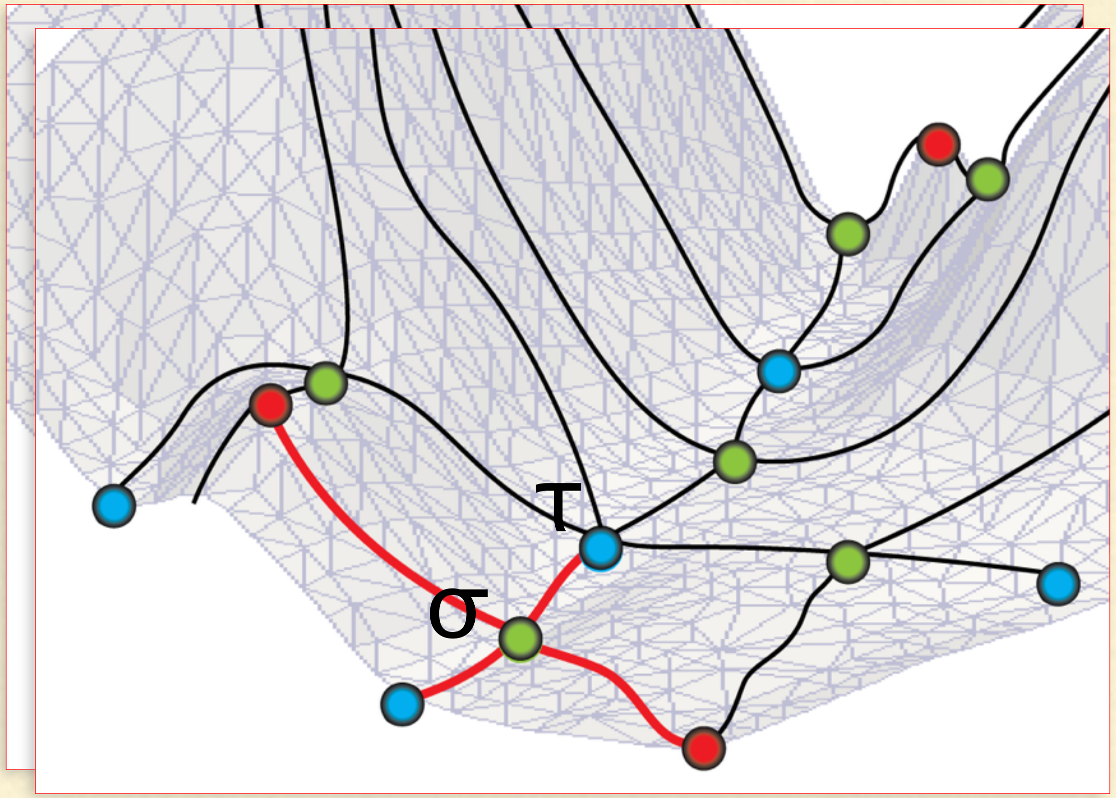

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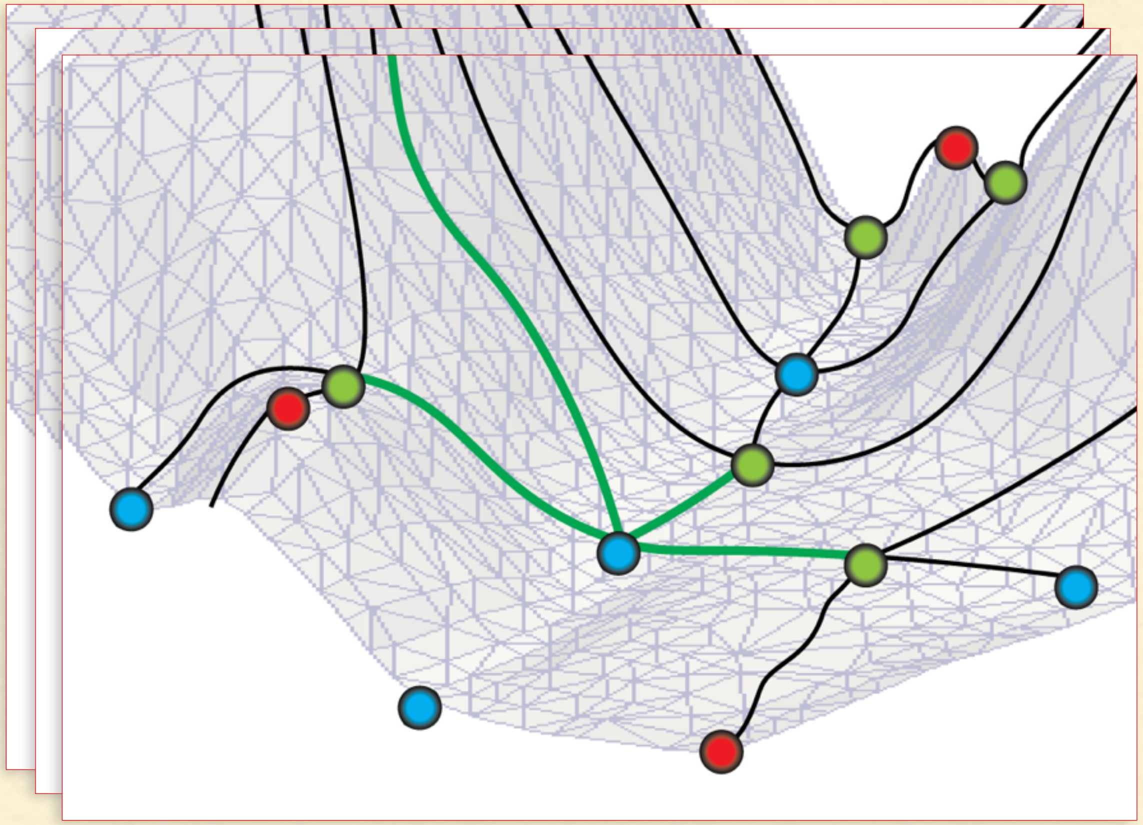

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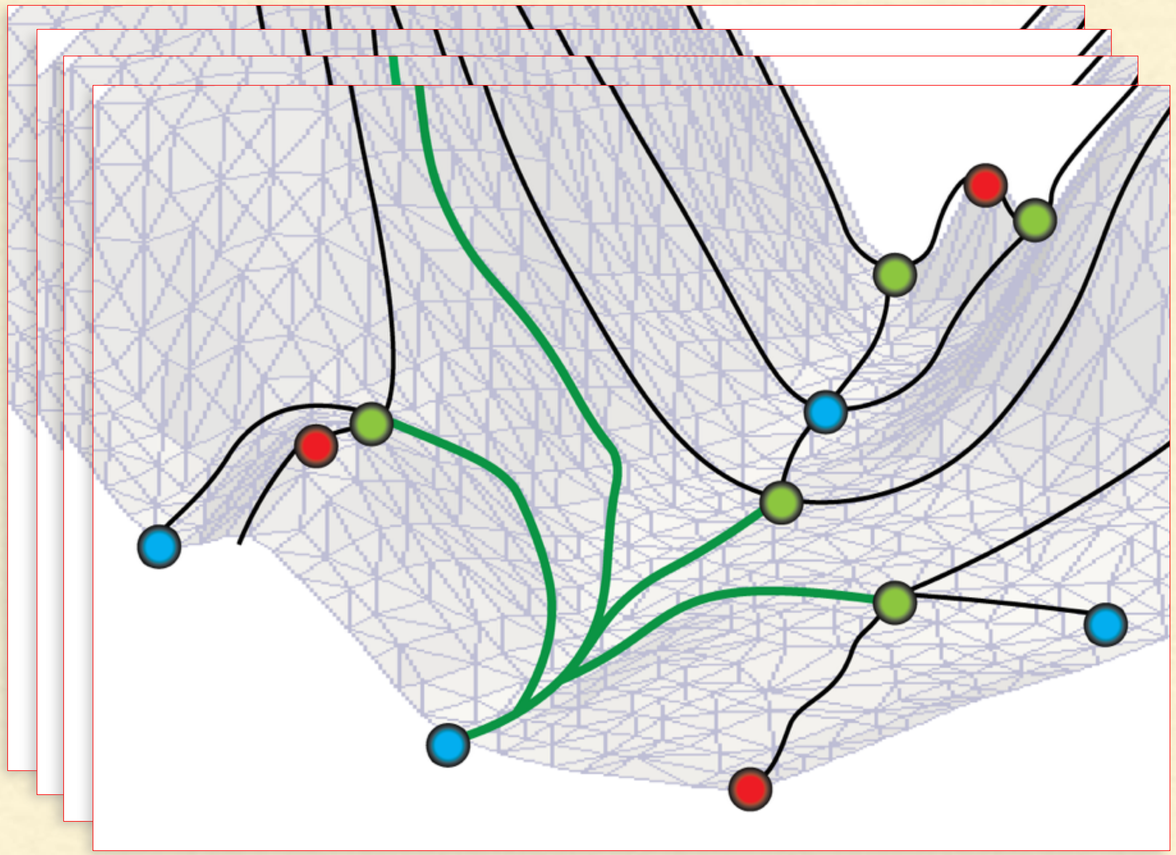


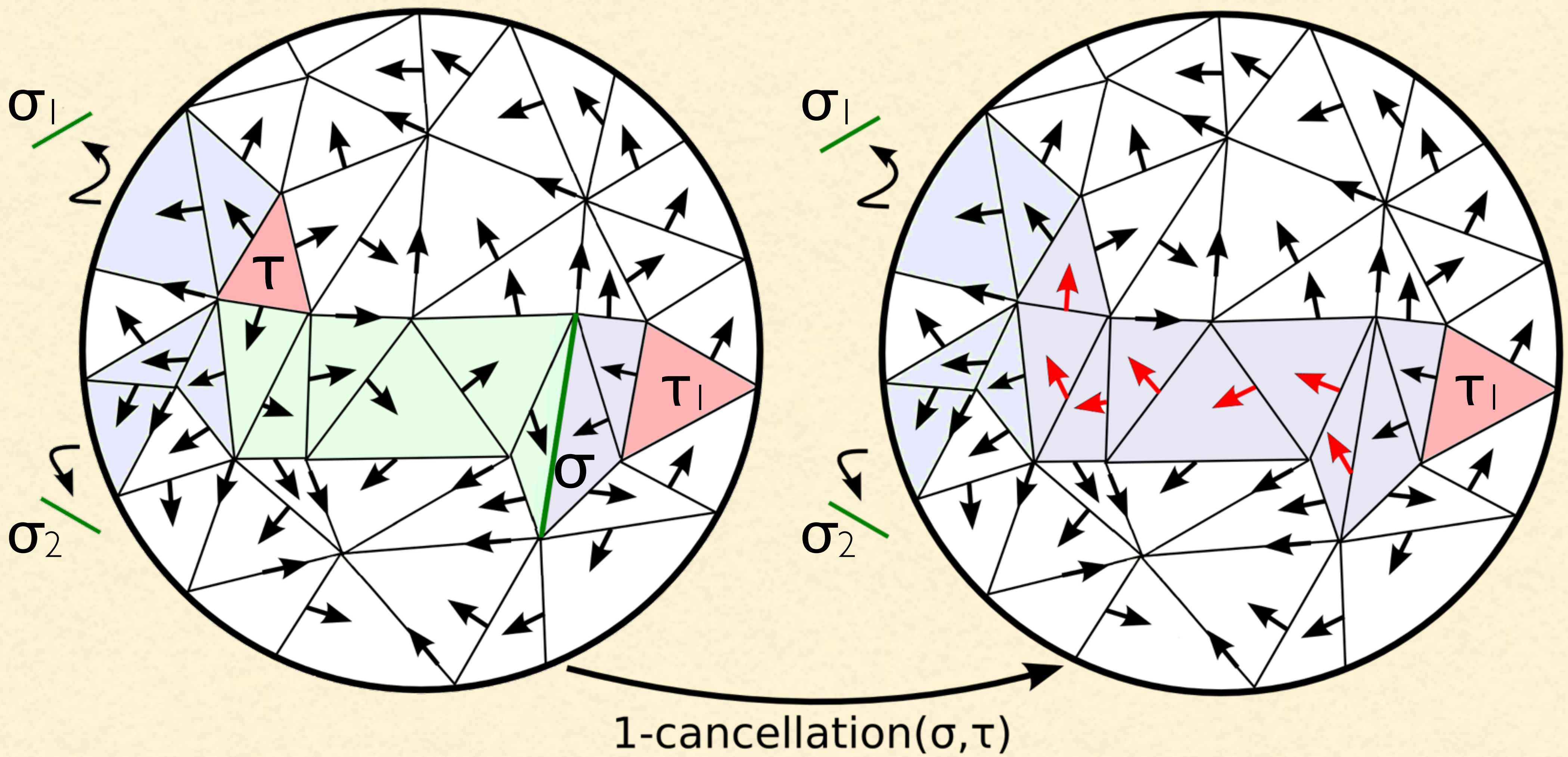
k-cancellation(σ,τ) removes a pair of critical simplices of index k and $k+1$ respectively under the assumption that

- σ and τ are connected by a unique *V*-path

CANCELLATION OPERATOR: GRADIENT-BASED REPRESENTATION

Effect of *k-cancellation*(**σ**,**τ**) on gradient-based representation:

- Reverse the gradient arrows along the unique *V*-path from **τ** to **σ**



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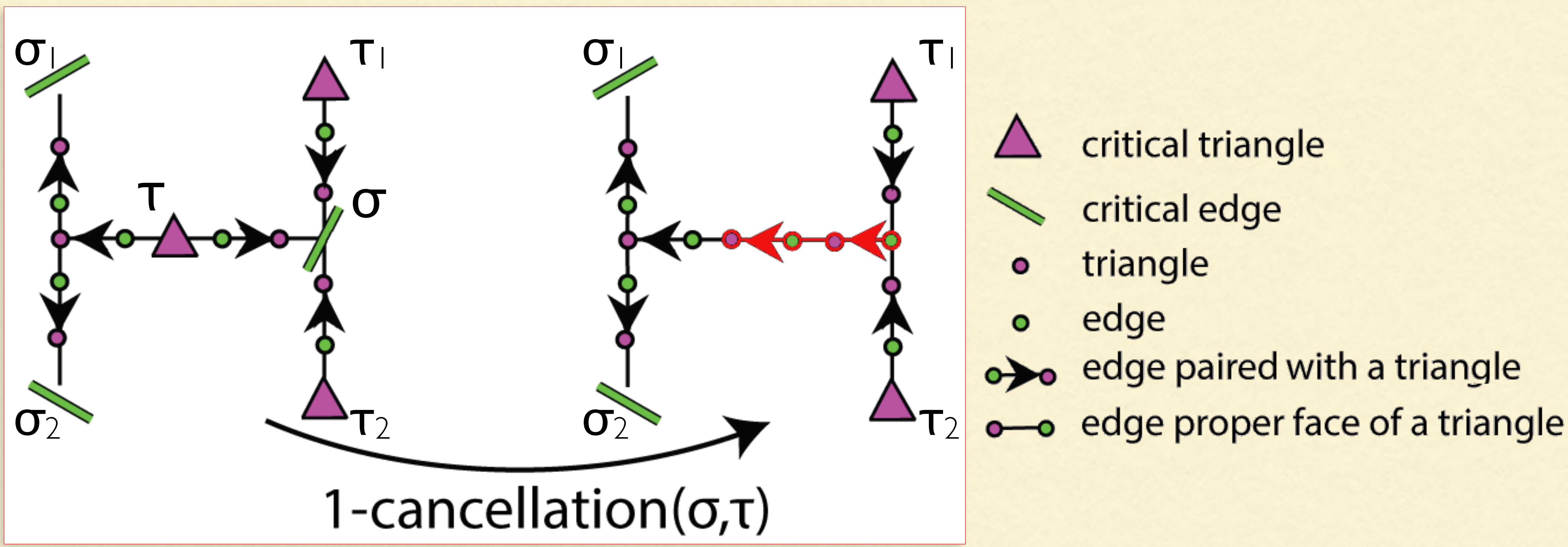
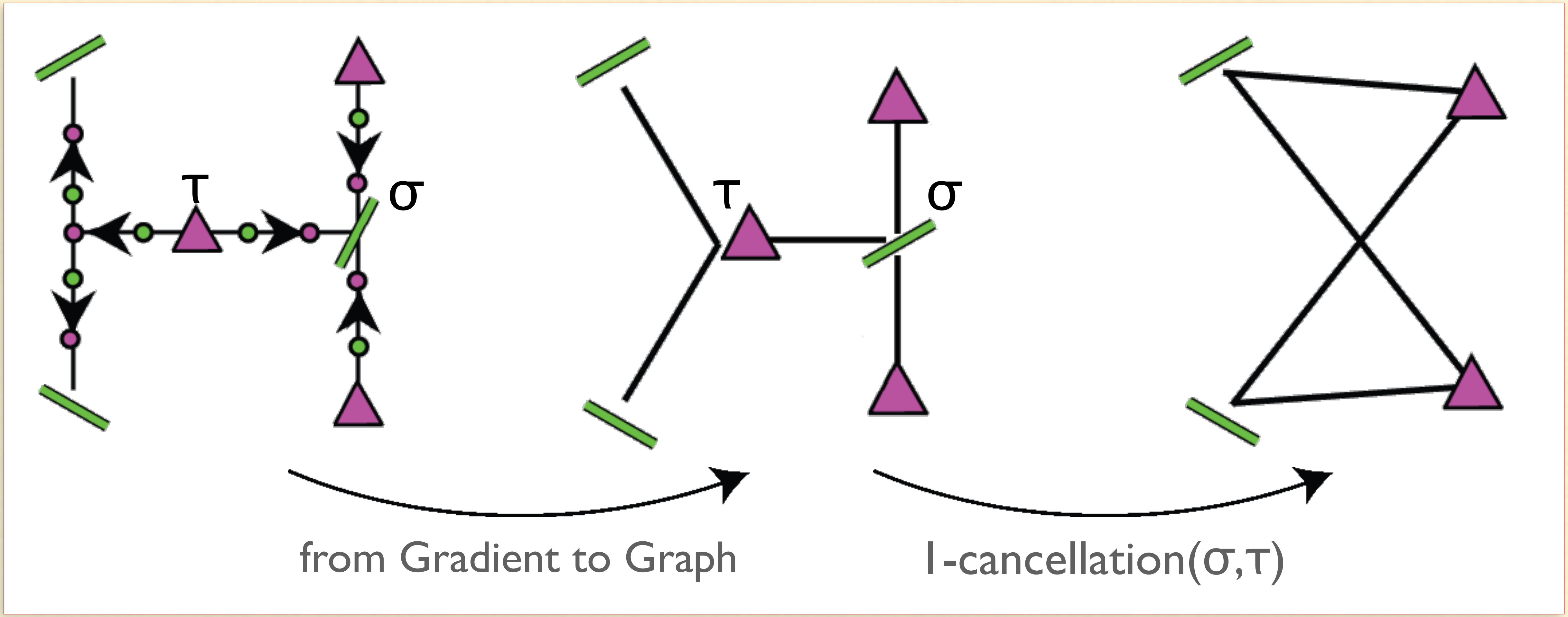


CANCELLATION OPERATOR: GRAPH-BASED REPRESENTATION

Effect of *k-cancellation*(**σ**,**τ**) on graph-based representation:

- Delete nodes **σ** and **τ** and all arcs incident in them
- Redirect arcs connected to **σ** and **τ** updating their weights



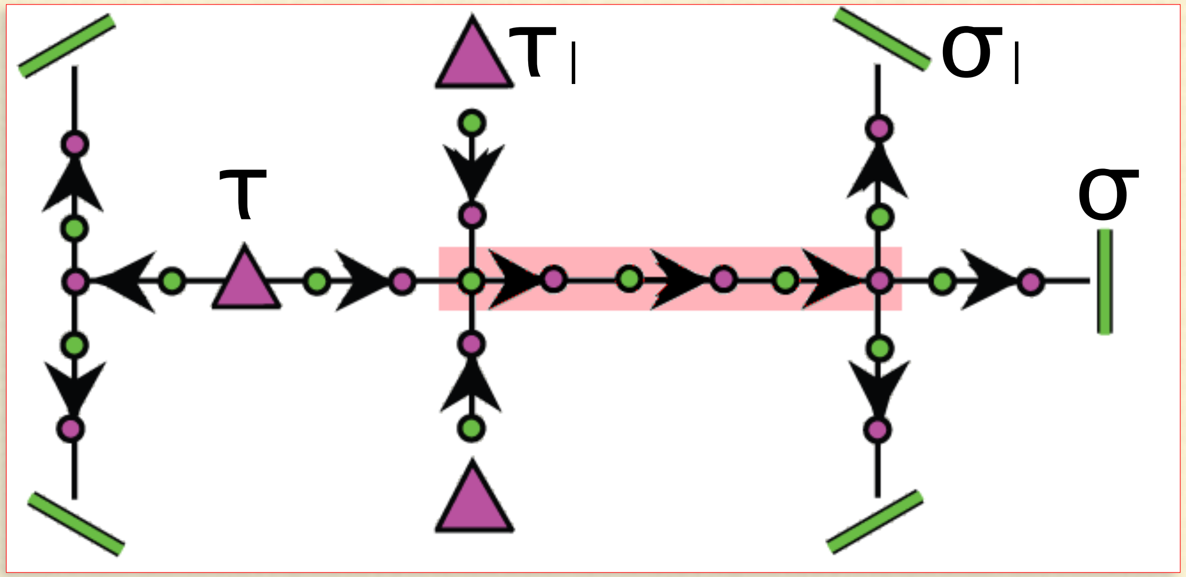
SIMPLIFYING MORSE COMPLEXES: TOPOLOGICAL INCONSISTENCIES

Up to dimension 2, the gradient-based and graph-based simplifications are equivalent

For complexes of higher dimensions, the two methods can produce different results [Günther et al. 2014]

Inconsistencies occur when *k-cancellation*(σ,τ) involves a **shared *V*-path**

- *V*-path in which different *V*-paths merge and split



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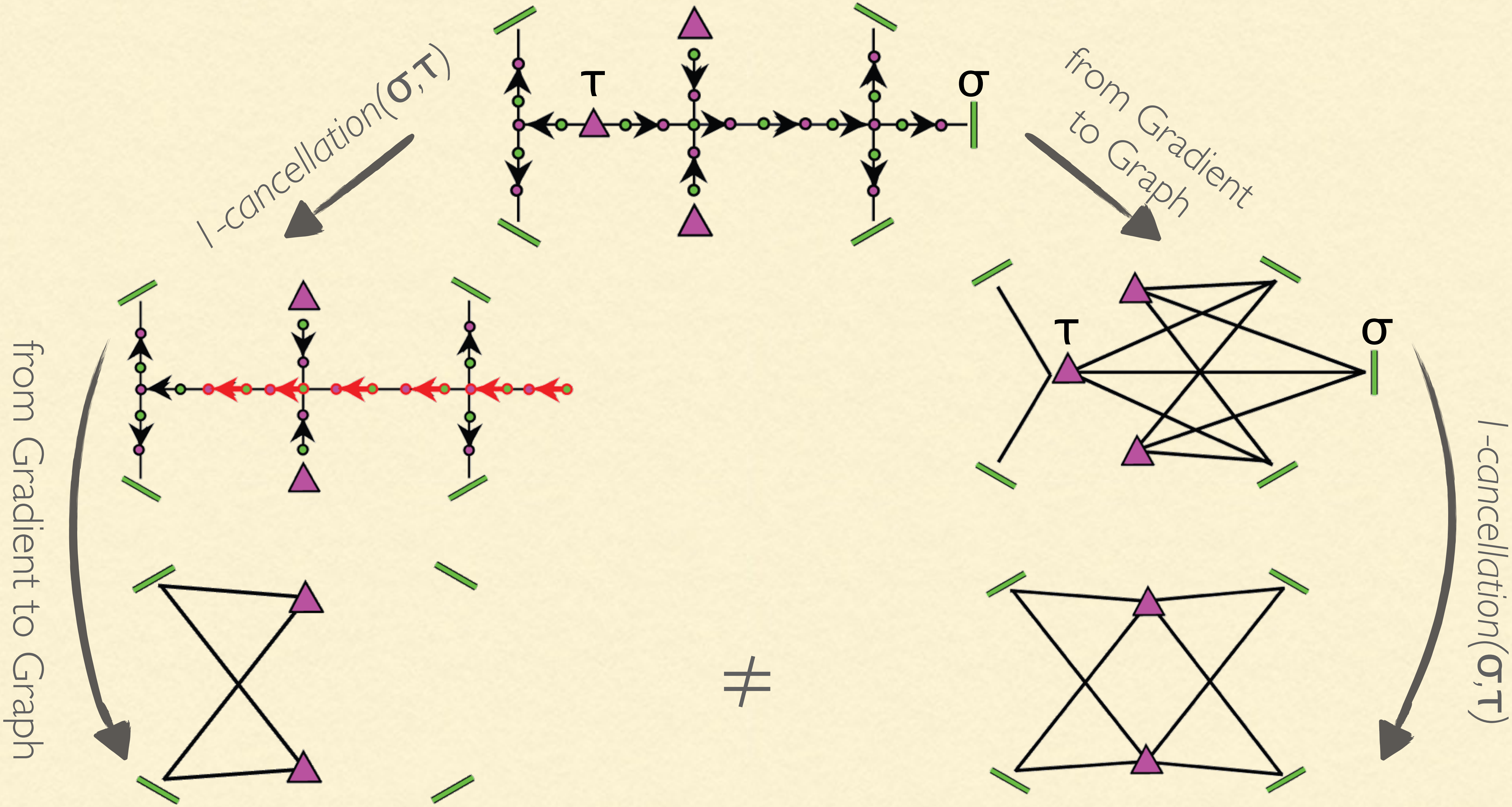
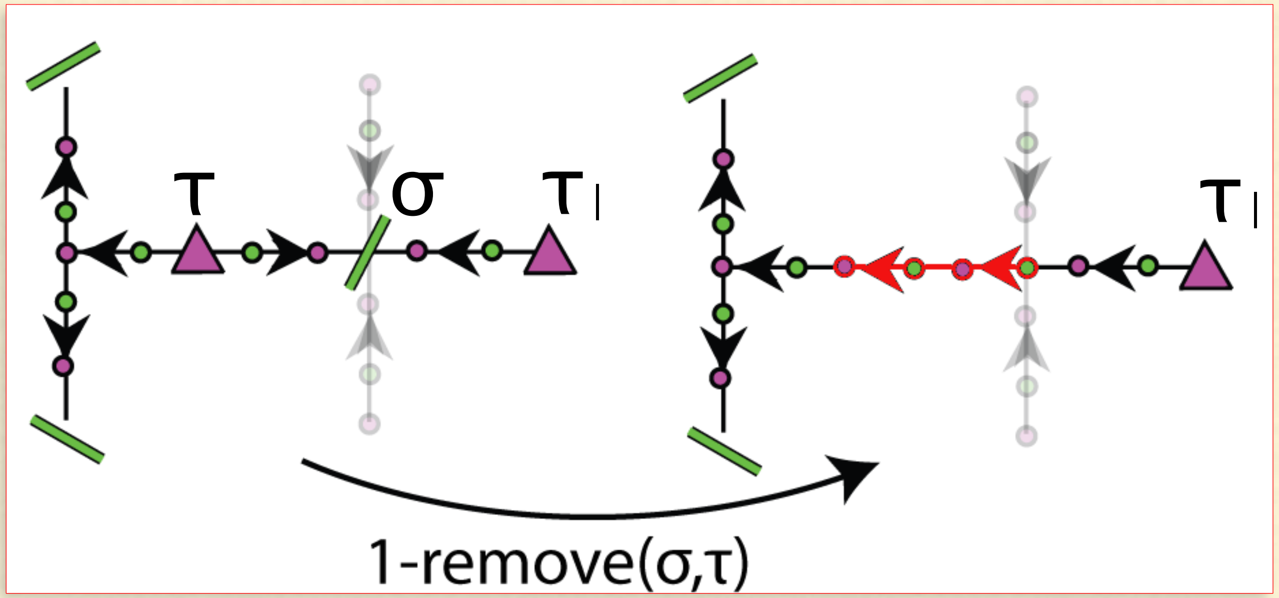


SIMPLIFYING MORSE COMPLEXES: REMOVE OPERATOR [ČOMIĆ ET AL. 2011]

k-remove(σ,τ) is a *k-cancellation*(σ,τ) in which at least one between the number of

- critical *k*-simplices connected to τ
- critical *(k+1)*-simplices connected to σ

is less or equal ≤ 2



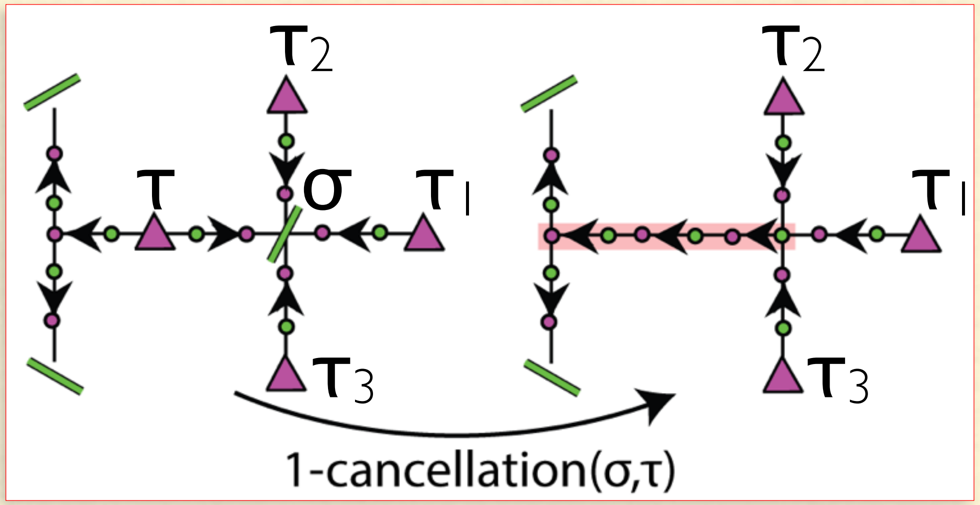
Analogously to the cancellation operator:
If a shared *V*-path is involved, *k-remove(σ,τ)* produces topological inconsistencies

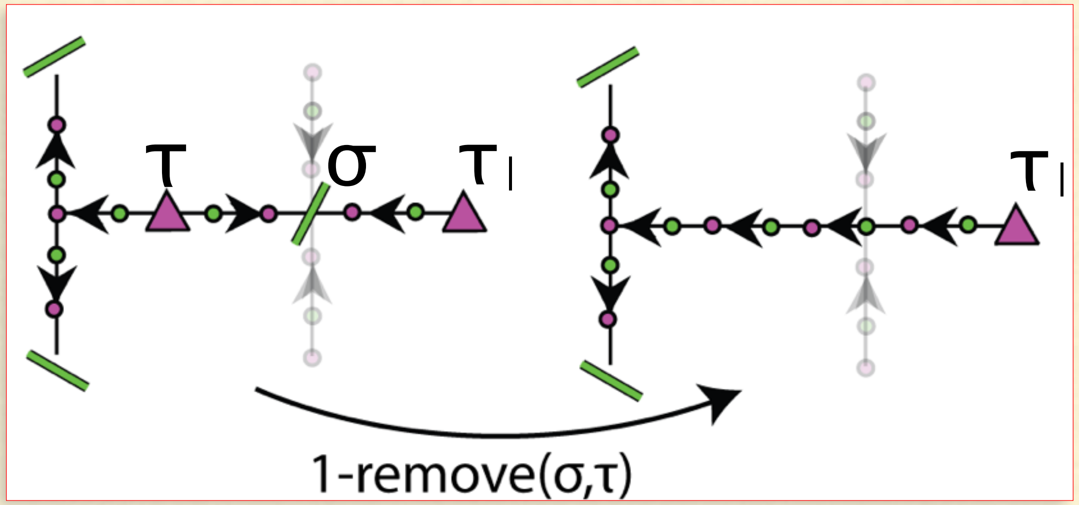
SIMPLIFYING MORSE COMPLEXES: REMOVE OPERATOR

Starting from a gradient free of shared *V*-path, remove operator does not introduce any shared *V*-path

Prop. Let *V* be a gradient free of shared of *V*-path. and *V'* the gradient obtained applying *k-cancellation*(σ,τ). Then,

V' does not contains any shared *V*-path $\Longleftrightarrow$ *k-cancellation*(σ,τ) is also a *k-remove*(σ,τ)



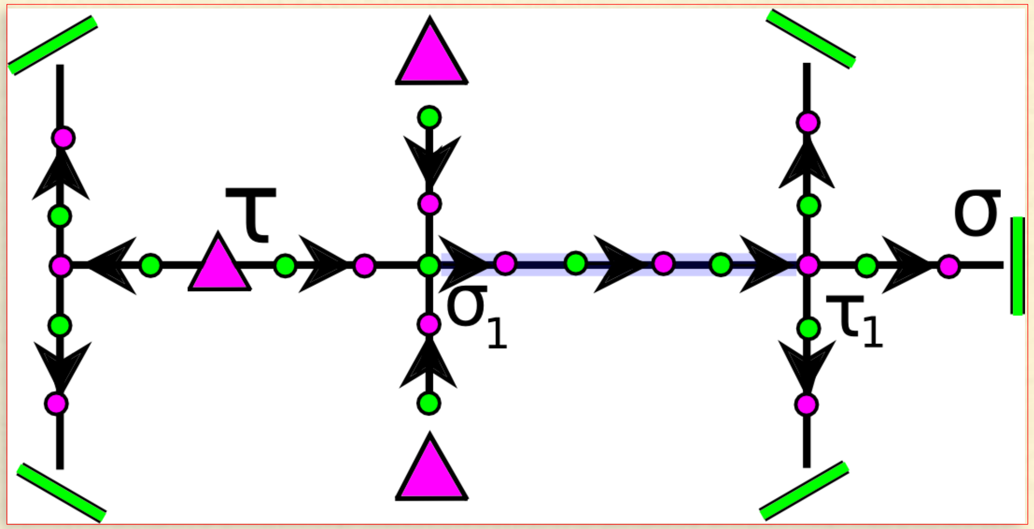


SIMPLIFYING MORSE COMPLEXES: SHARED *V*-PATH DISAMBIGUATION

We propose a **preprocessing step to untie the shared *V*-paths** in a simplicial complex Σ endowed with a gradient *V*

The steps of the shared *V*-path disambiguation algorithm are the following:

- Navigate the gradient from *k*- to *(k+1)*-saddles to **identify shared *V*-paths**



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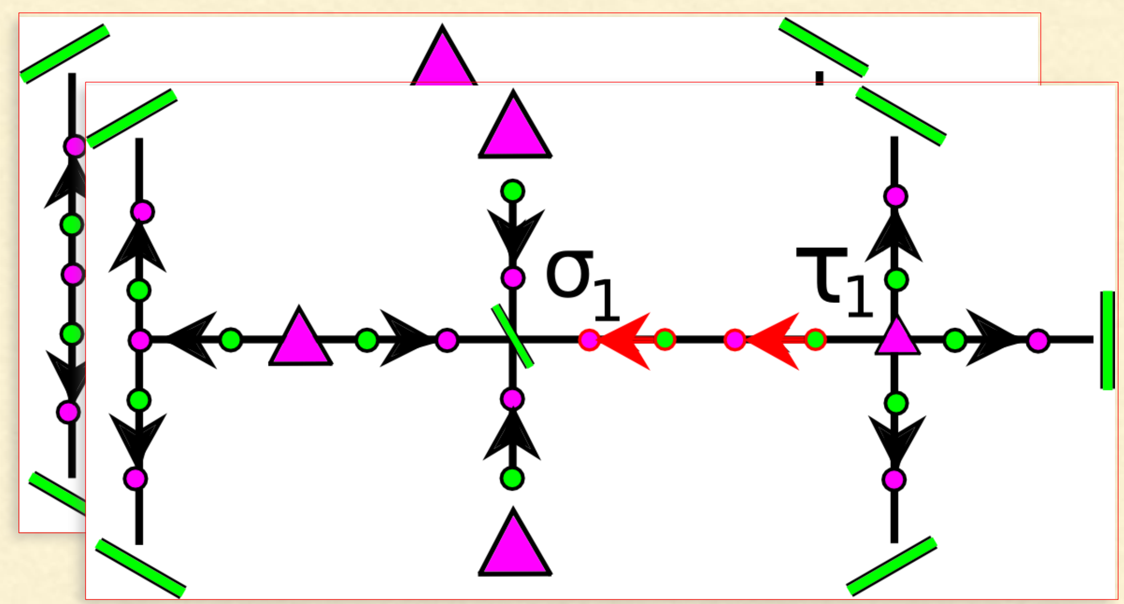

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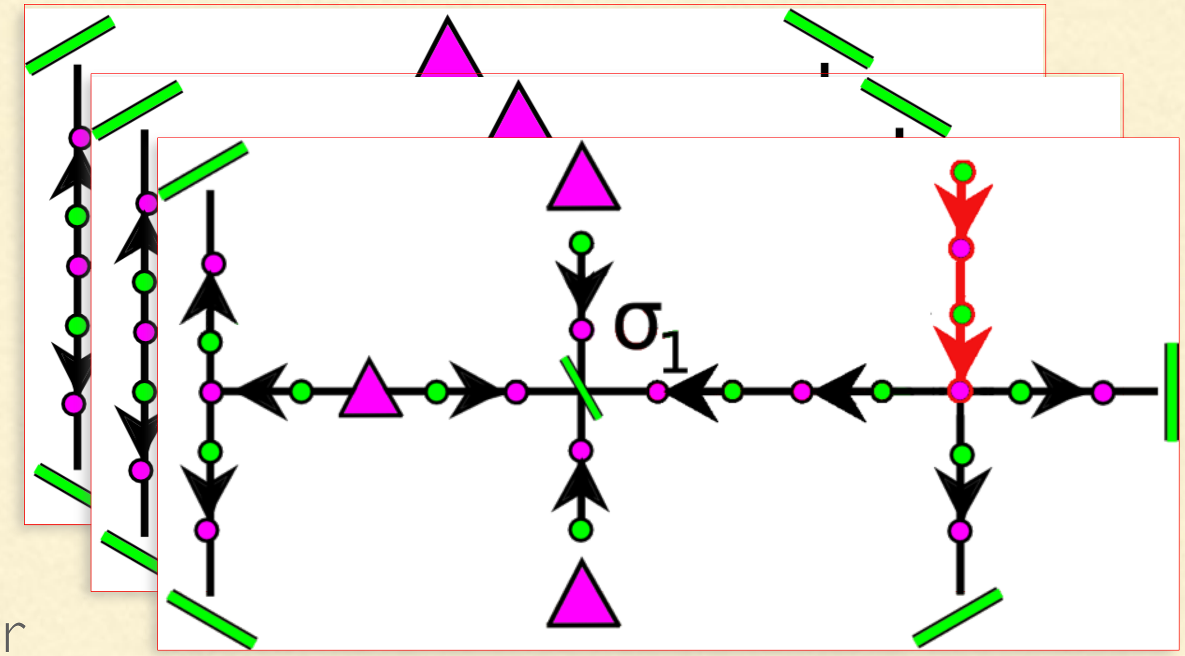
- **Introduce a pair of dummy critical simplices** σ_1,τ_1 thanks to the undo of *k-cancellation*(σ_1,τ_1)
- Perform a simplification step to **remove all the dummy critical simplices** by using remove operator



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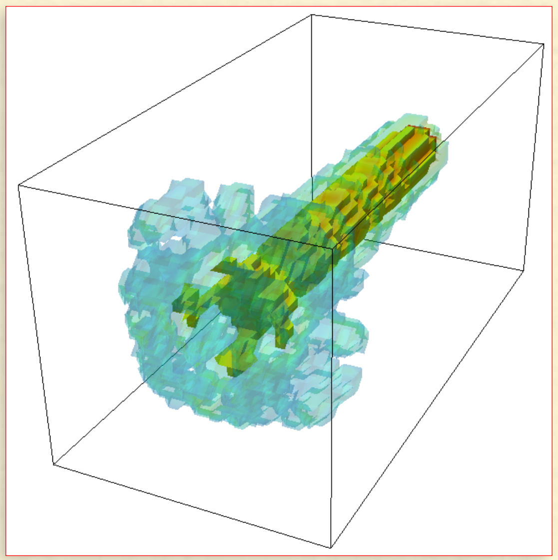
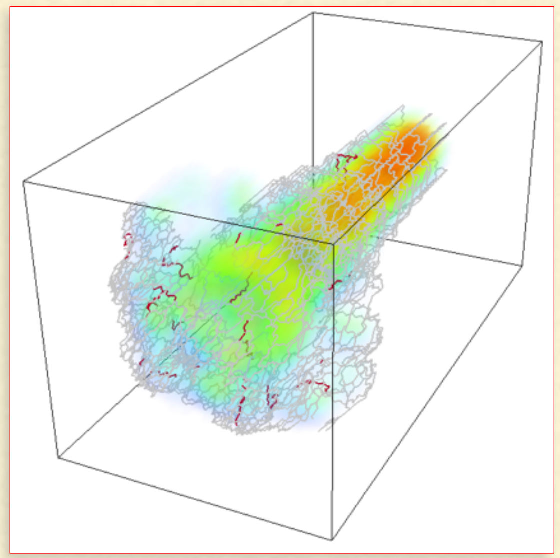
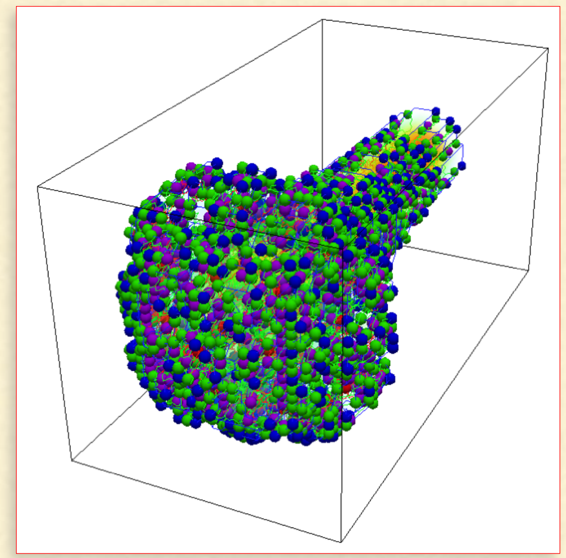
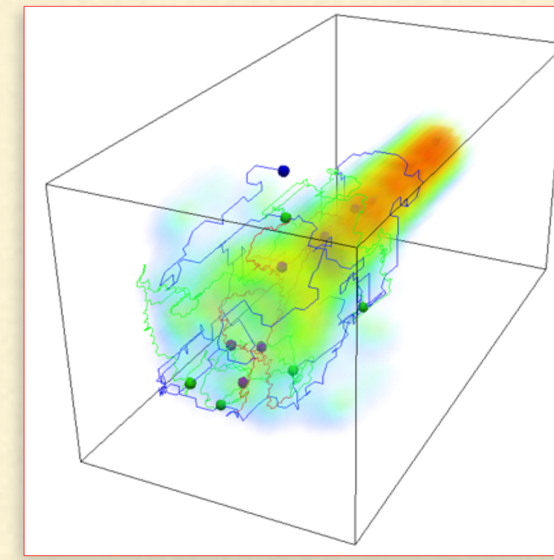
SIMPLIFICATION ALGORITHM

We have developed and implemented for unstructured tetrahedral meshes a **topologically-consistent simplification algorithm** consisting of

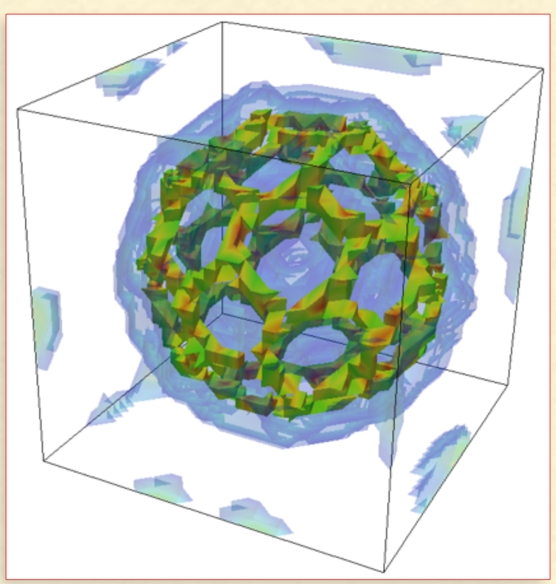
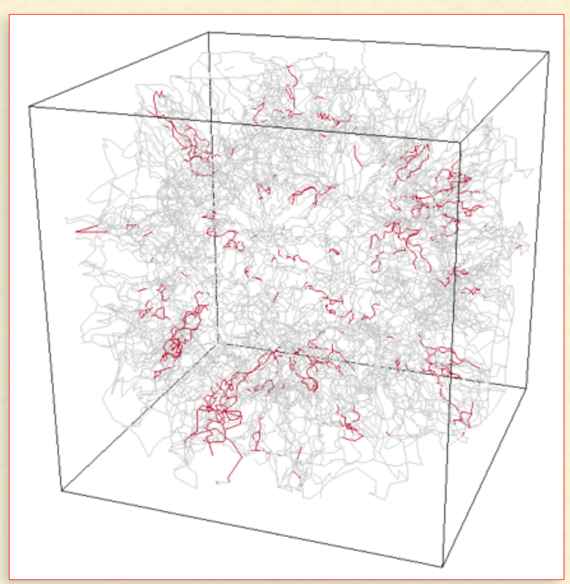
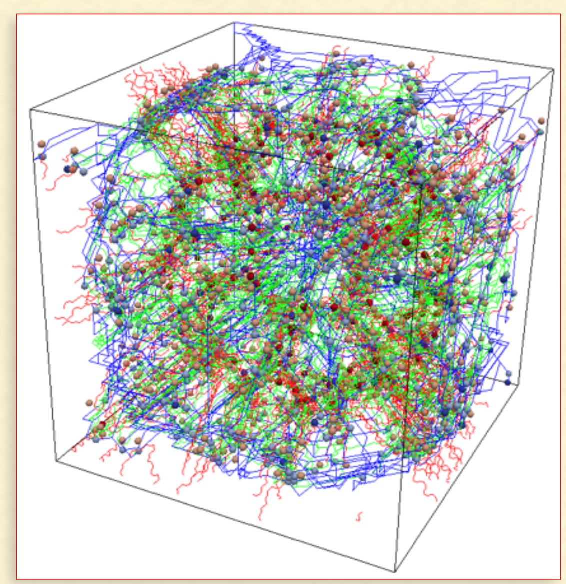
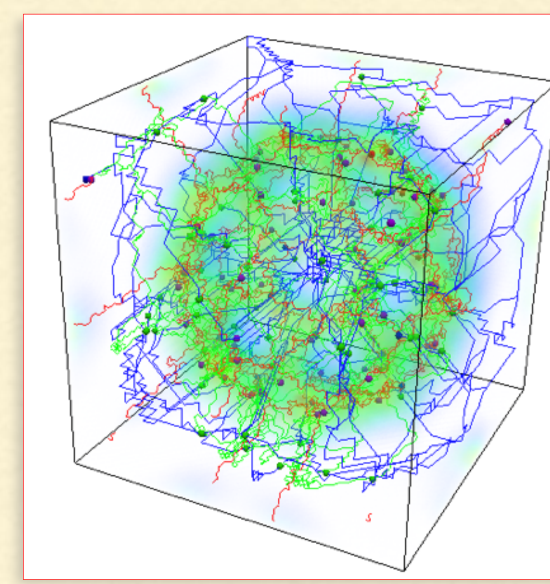
- **Preprocessing step**: shared *V*-path disambiguation algorithm
- **Simplification algorithm** based on remove operator
    - remove operators are applied in ascending order of persistence

Data structure for representing Morse complexes:
Discrete Morse Incidence Graph (DMIG)

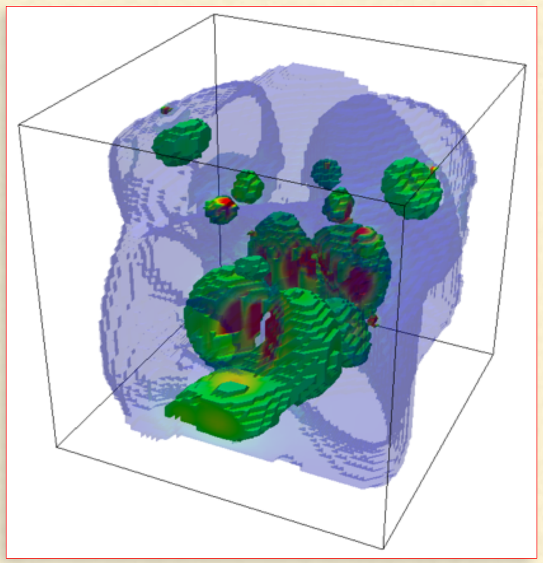
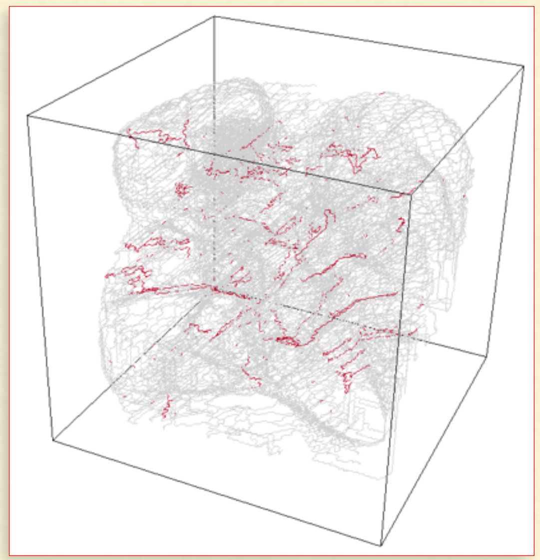
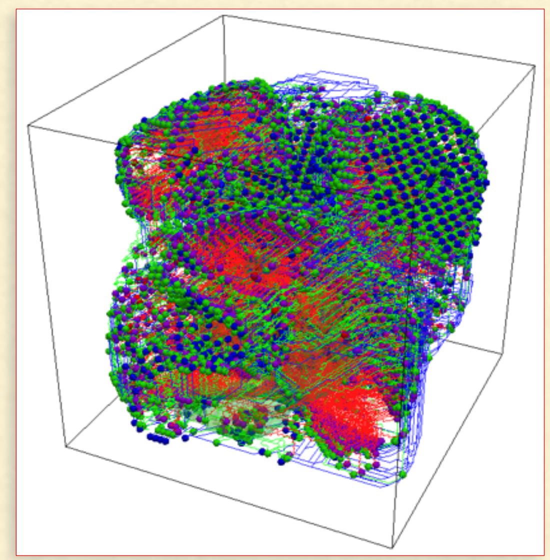
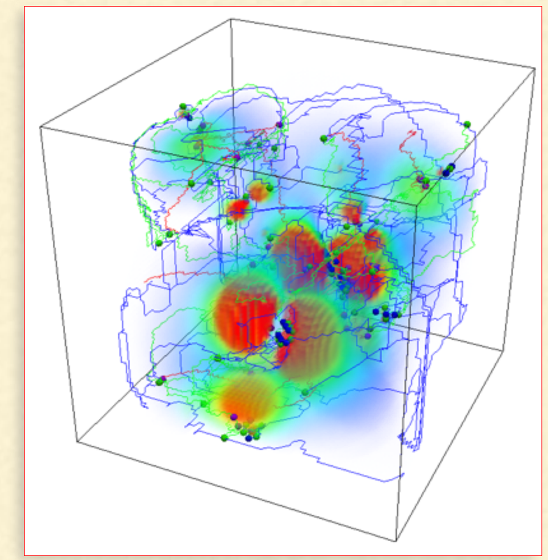
SIMPLIFICATION ALGORITHM: EXPERIMENTAL RESULTS

Original scalar field | Shared *V*-paths | Original DMIG | Simplified DMIG

SIMPLIFICATION ALGORITHM: EXPERIMENTAL RESULTS

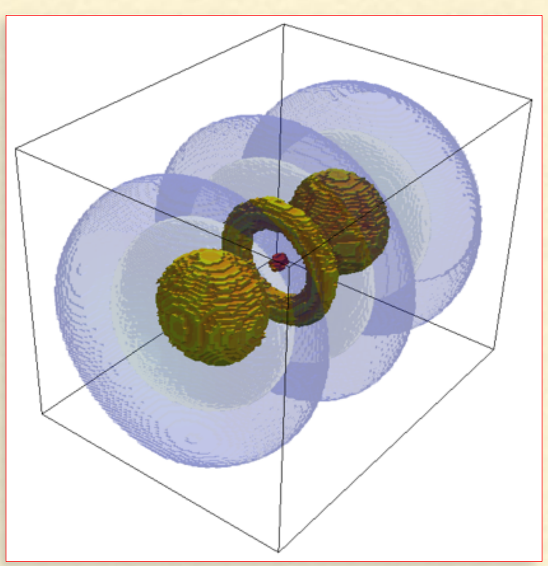
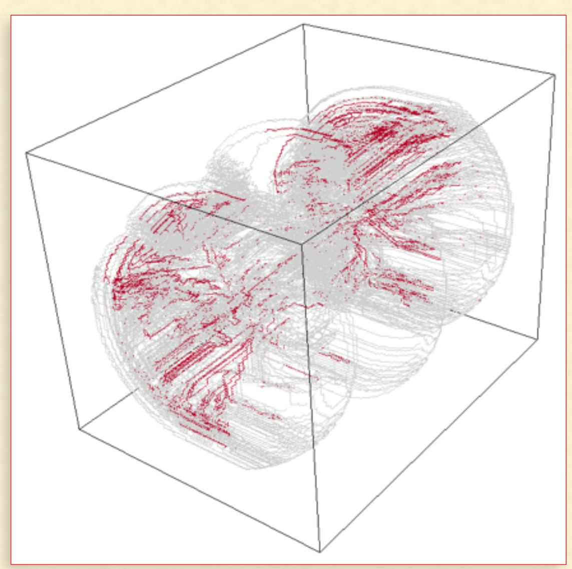
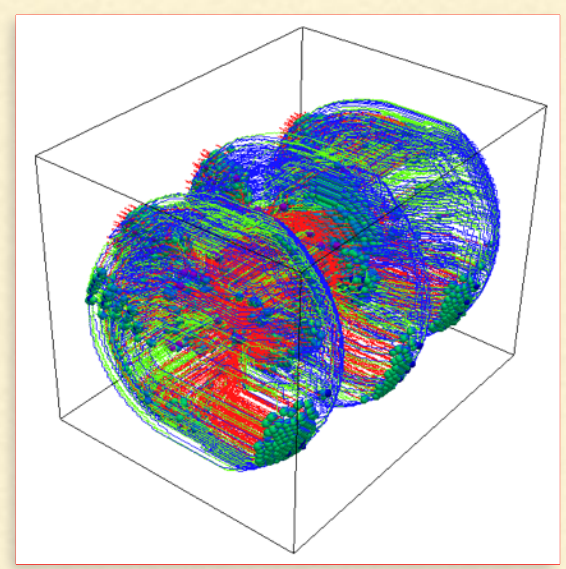
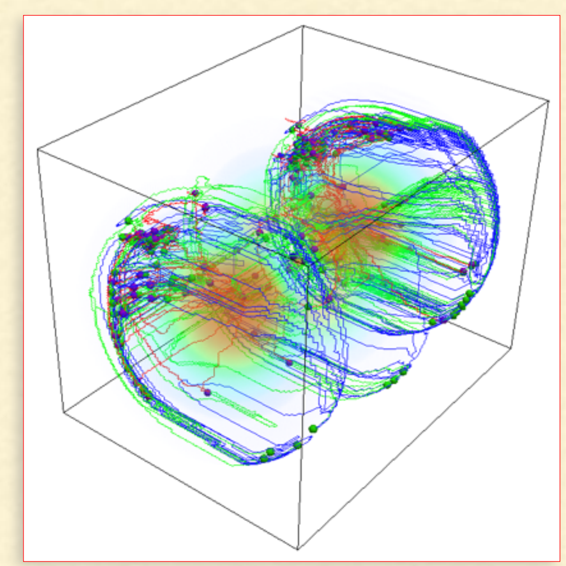





Original scalar field

Shared *V*-paths

Original DMIG

Simplified DMIG

SIMPLIFICATION ALGORITHM: EXPERIMENTAL RESULTS

Evaluation of the preprocessing step and of the remove-based simplification

Timings:

- **Preprocessing:** from 0.65 s up to 24.1 min
- **Simplification:** from 4.13 s up to 24.3 min

Dataset	*Size*	$\lvert\Sigma_0\rvert$	$\lvert\Sigma_3\rvert$	#*C*
BUCKY	32^3	32K	0.17M	2K
FUEL	64^3	13K	0.06M	2.7K
SILICIUM	98*x*34*x*34	66K	0.36M	2.1K
NEGHIP	64^3	0.12M	0.64M	12.6K
SHOCKWAVE	64*x*64*x*512	1.2M	7M	1.1K
BLUNT	256*x*128*x*64	1.0M	6M	11.2K
HYDROGEN	128^3	0.6M	3.9M	15.1K

Dummy critical simplices introduced: 2-13% of the total number of critical simplices

Maximum amount of memory: from 0.05 to 2.2 GB

CURRENT AND FUTURE WORK

We have developed and implemented a new compact and topologically-consistent algorithm for a morphological simplification of Morse complexes

The algorithm proposed is a basis tool for

- Simplification algorithm performing both morphological and **geometric** operations (through edge contraction) concurrently
  - done for the 2D case [Fellegara et al. 2014]
- A topological **multi-resolution** model

We plan to develop a **distributed approach** for the simplification algorithm by using a **stellar tree** data structure [Fellegara 2015]