A Hierarchical Approach to Topological Shape Analysis

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1 Motivations

Our interest - applying tools from algebraic topology (homology) to the description and analysis of shapes

- defined by finite sets of points organized to form discrete structures, like cells or simplicial complexes
- here we use cell complexes

Issues both theoretical and computational

- size of the data sets: often huge collections of unorganized sets of points
- dimension of the data sets: high-dimensional data

Computational problems in using topological tools

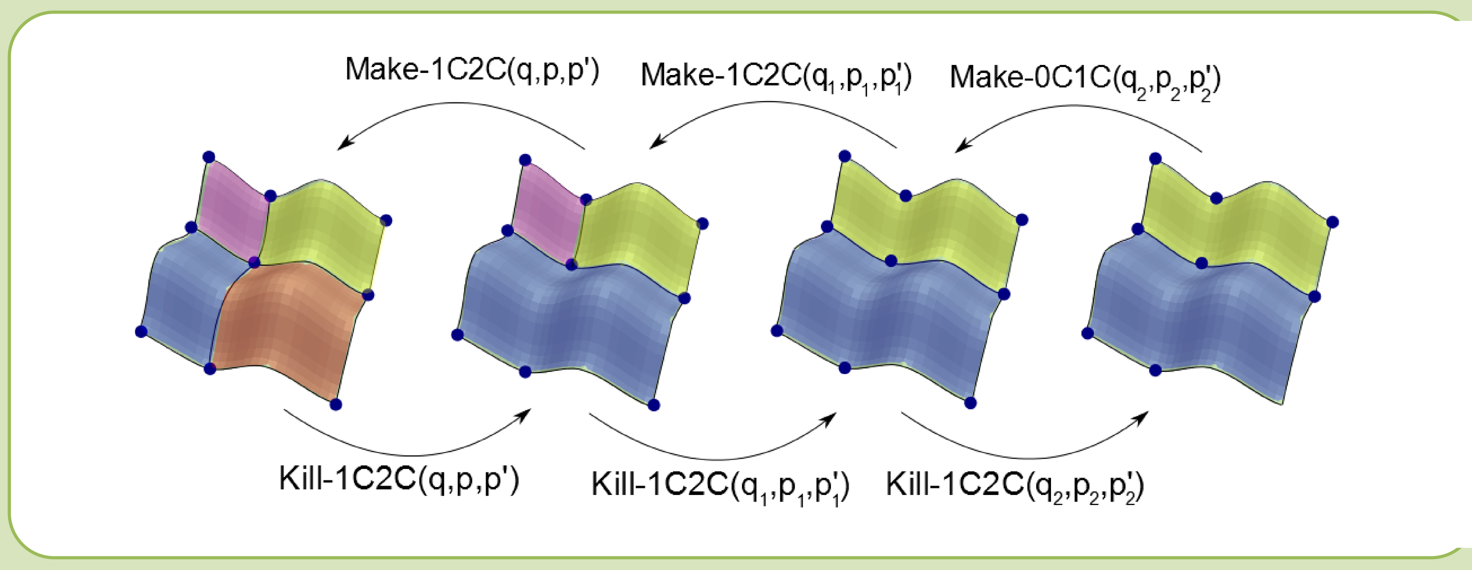
- high computational costs
- high storage costs

Our approach here: use of hierarchical, multi-resolution representations

2 Hierarchical Cell Complex (HCC)

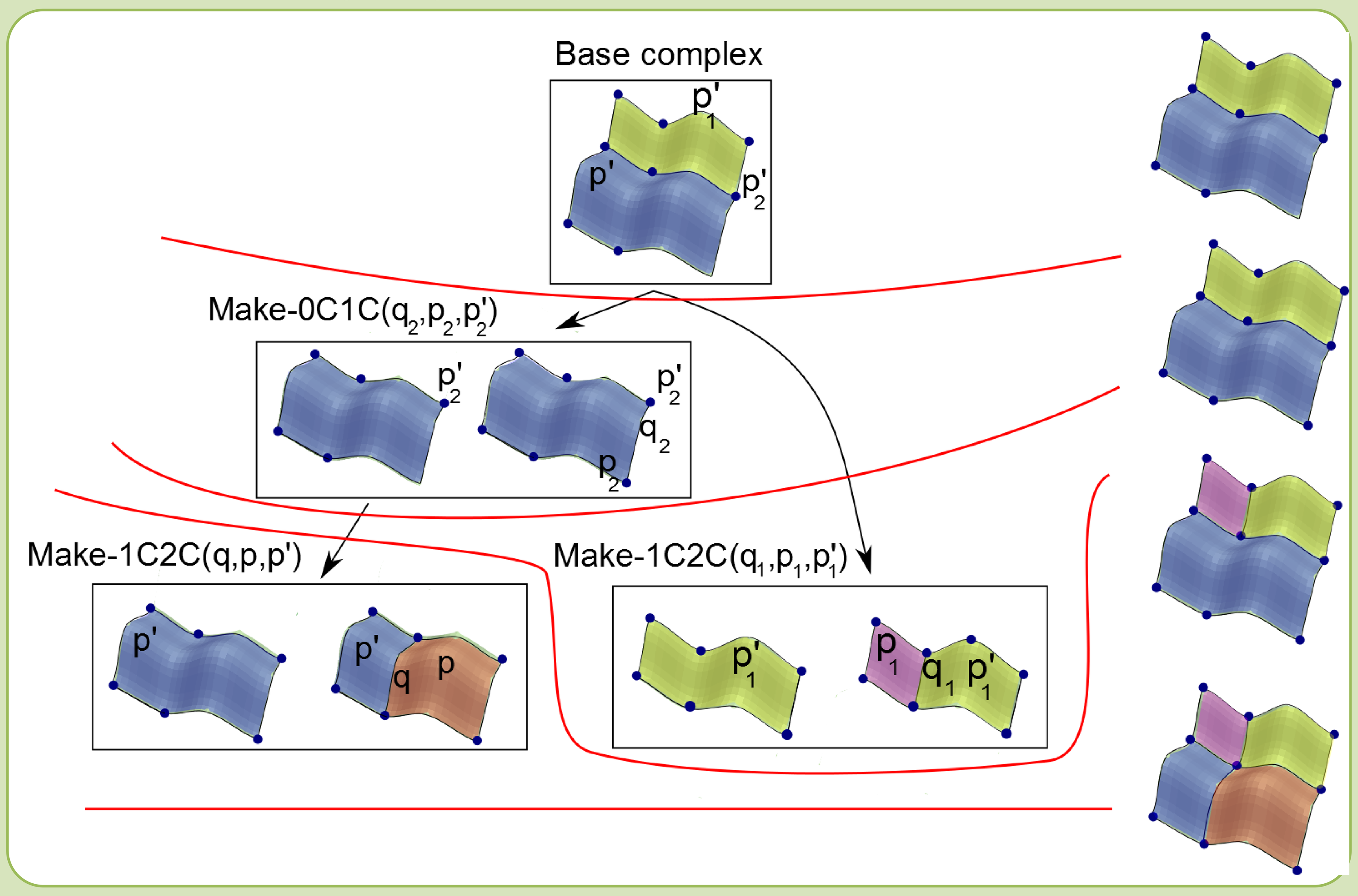
Based on the **simplification** of a given cell complex *X* by iteratively applying atomic **homology-preserving** operators (*Kill-iCell-(i+1)Cell*)

- this generates a **sequence *S*** of simplifications



An ***HCC*** consists of *three components (X', M, R):*

- ***base complex*** *X'* obtained from *X* by applying the sequence *S* of simplifications
- set *M* of **refinements** *(Make-iCell-(i+1)Cell),* inverse of the operators in sequence *S*
- ***direct dependency relation R*** between pairs of refinements, described as a *Directed Acyclic Graph (DAG):*
  - a ***refinement*** *μ=Make-iCell-(i+1)Cell (q,p) directly **depends*** on a refinement *μ** if and only if μ* creates a cell that will be in the immediate boundary or co-boundary of *p* or *q*



- The transitive closure of the direct dependency relation can be shown to be a ***partial order***

In an *HCC (X',M,R),* there is a ***one-to-one correspondence*** between

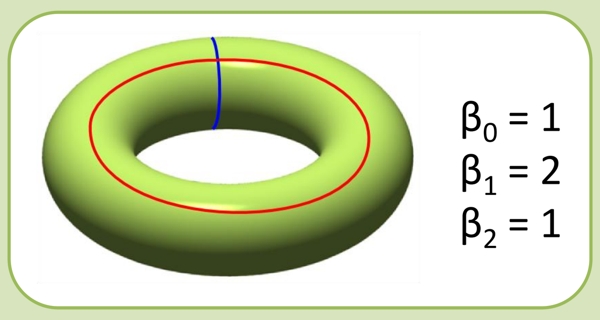
- the subsets of refinements in *M*, which are ***closed*** w.r.t. respect to the dependency relation, and the complex which can be extracted from it

Extraction process (***selective refinement***):

- top-down traversal of the DAG with iterative application of *Make-iCell-(i+1)Cell* refinements to the base complex *X'*

3 Homology Computation on the HCC

We are interested **in computing the homology groups** H_i *(X;* $\mathbb{Z}_2$*)* of a cell complex *X* with coefficients in $\mathbb{Z}_2$



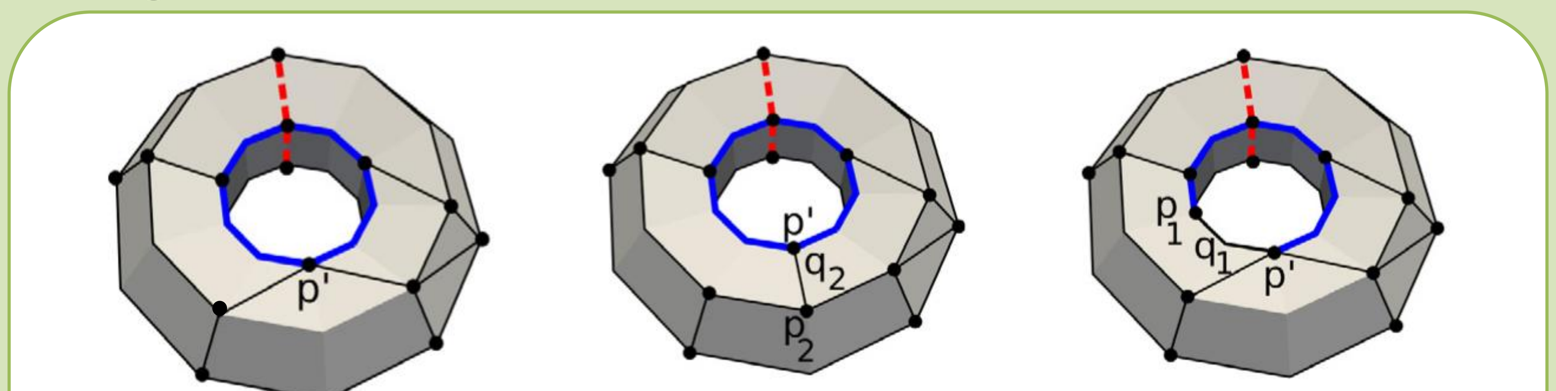
- the Betti numbers β_i's of *X*
  - β_i measures the number of independent non-bounding *i*-cycles in *X* (*i*-holes)
- the homology generators of degree *i*: generators of the $\mathbb{Z}_2$-vector space H_i *(X;* $\mathbb{Z}_2$*)*

All complexes, which can be extracted from an *HCC(X',M,R),* have the same homology of the original complex *X*

- **Smith Normal Form** (SNF) reduction used to compute homology on the base complex *X'.*

Computing generators during selective refinement

- application of *Make-iCell(i+1)Cell* only affects generators of degree *i+1*



Original complex **not involving generators** **involving generators**

4 Experimental Evaluation

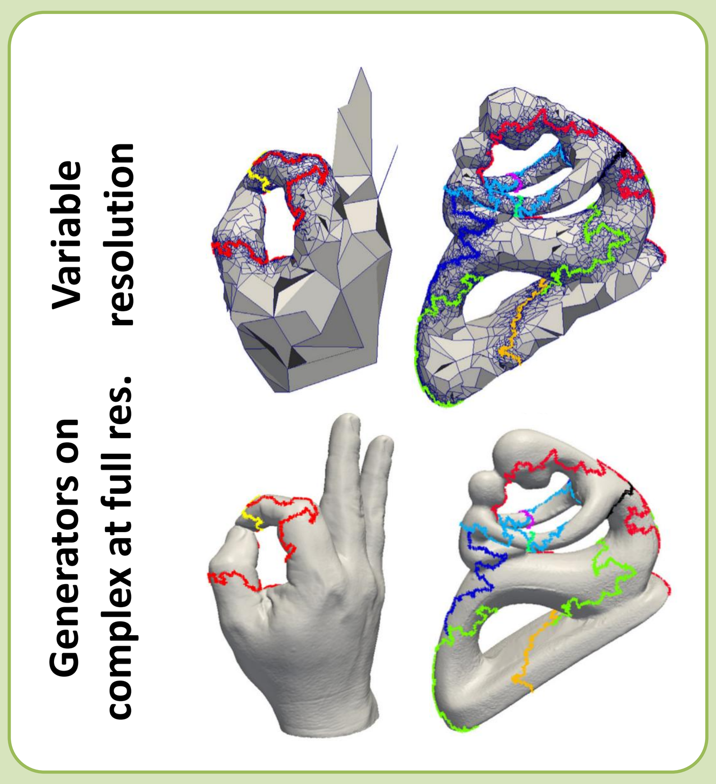
Computation of homology on the base complex plus refining all the generators to full resolution:

- min 0.15 seconds – max 83.3 seconds
- on the smallest complex (*Genus3*- 40K cells) 2.6 hours for homology computation

Extraction at *uniform* and *variable* resolutions:

- variable resolution: maximum detail only in the neighborhood of specific homology generators

Dataset	*SNF* Time	Tot Ref Time	Uniform Ref.	Uniform Time	Generators Ref.	Generators Time
Genus3	9.2×10^{-5}s	0.15s	4K 10K 16K	0.03s 0.07s 0.12s	5K	0.03s
Fertility	8.3×10^{-5}s	9.31s	144K 362K 579K	1.8s 4.6s 7.52s	68K	1.48s
Hand	9.8×10^{-5}s	14.9s	200K 500K 800K	2.6s 6.8s 11.2s	19K	1.6s
Buddha	0.02s	23.7s	320K 800K 1.2M	0.5s 4.3s 19.2s	162K	3.6s
Skull	0.007s	6.4s	75K 187K 299K	1.0s 2.9s 5.0s	191K	2.6s
Fert-Solid	8.8s	74.5s	1.2M 3.1M 4.9M	7.5s 29.1s 69.3s	267K	10.9s



Datasets – triangle and tetrahedral complexes (between 40K and 6.2M cells): storage cost between 4.8 and 980 MB.

L. Čomić, L. De Floriani, and F. Iuricich. Multi-resolution cell complexes based on homology-preserving Euler operators. In *17th IAPR International Conference on Discrete Geometry for Computer Imaginary (DGCI)*, pages 323-334, Seville, Spain, March 2013.

L. Čomić, L. De Floriani, U. Fugacci and F. Iuricich. Topological modifications and hierarchical representation of cell complexes in arbitrary dimensions. *In Computer Vision and Image Understanding (CVIU)* Vol.121 pages. 2-12, 2014.