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5th International Workshop on Computational Topology in Image Context

EFFICIENT COMPUTATION OF SIMPLICIAL HOMOLOGY THROUGH ACYCLIC MATCHING

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MOTIVATION

Apply *topological methods* to the *description* and the *analysis of shapes*

We are interested in:

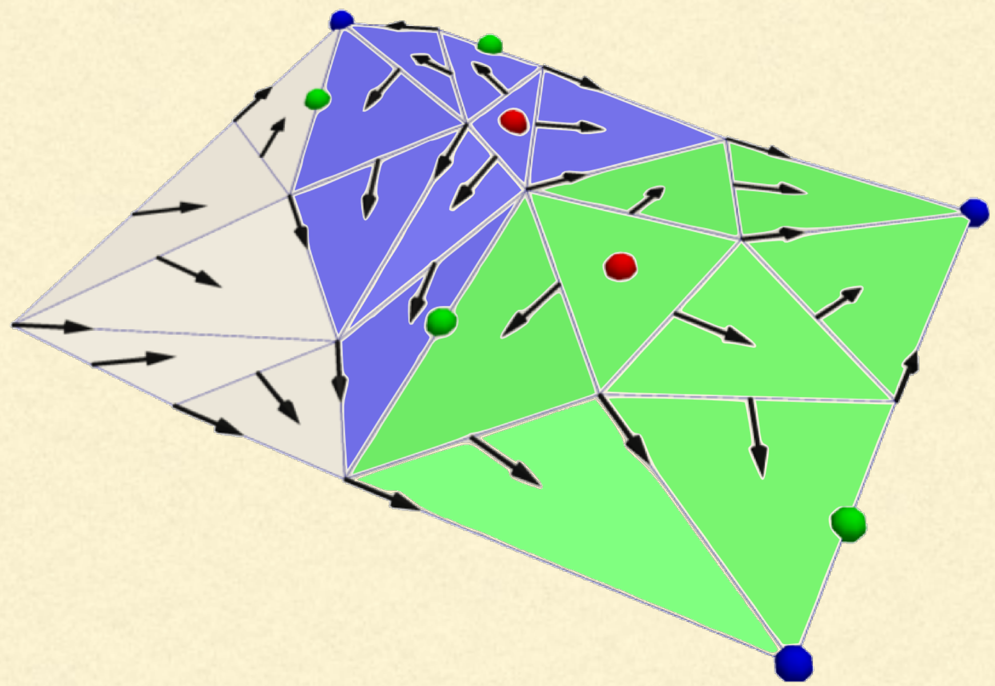
- *Large-size* data
- *High-dimensional* data

Main tool: Simplicial Homology

OUTLINE

- **Background notions**
  - Discrete Morse theory
  - Reductions and coreductions
- **Discrete Morse theory through reductions and coreductions**
  - Reduction-based and coreduction-based approach
  - Equivalence of the two approaches
  - Interleaving reductions and coreductions
- **Our algorithm**
  - Efficient encoding for the simplicial complex
  - Efficient encoding for the gradient vector field
- **Conclusions**

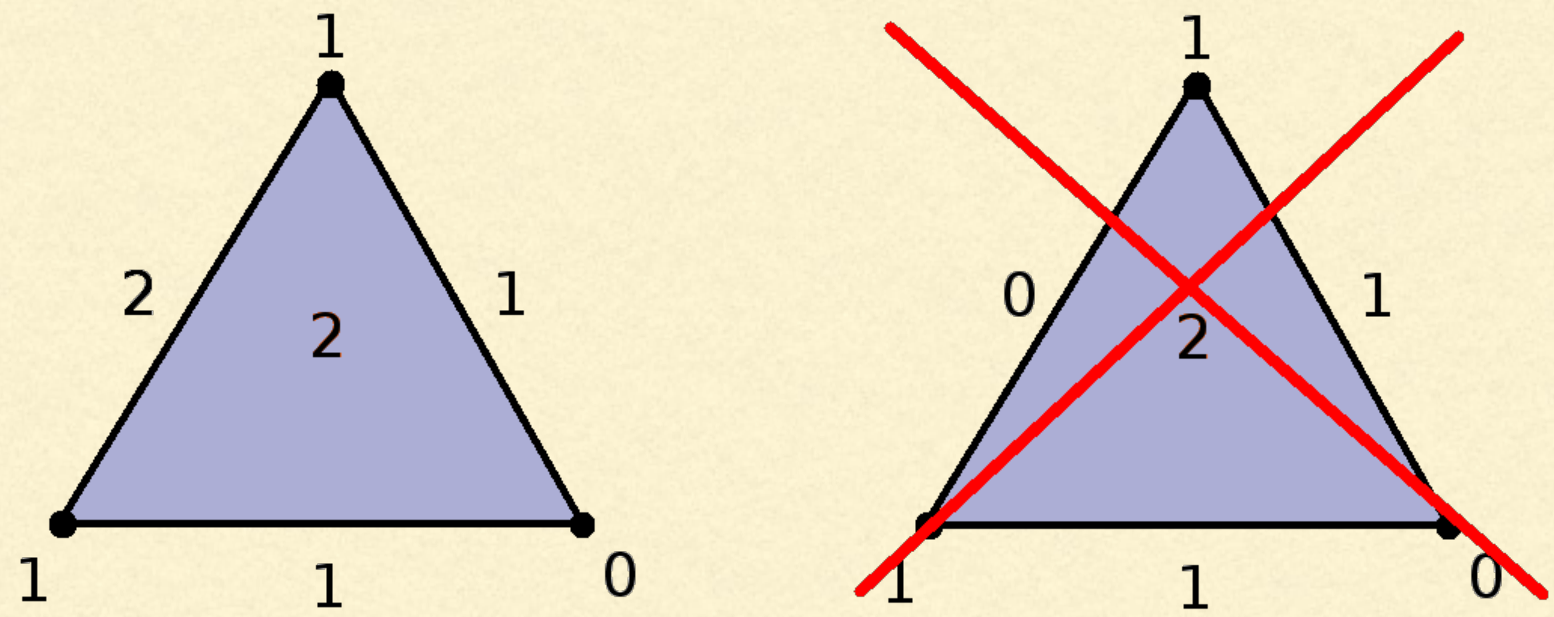
DISCRETE MORSE THEORY
[FORMAN 1998]



- *Combinatorial counterpart* of Morse theory [Milnor 1963]
- Introduced for *CW complexes*
- Gives a compact *homology-equivalent model* for a shape
- Provides *topological invariants* from a gradient vector field

DISCRETE MORSE FUNCTION

Let Σ simplicial complex

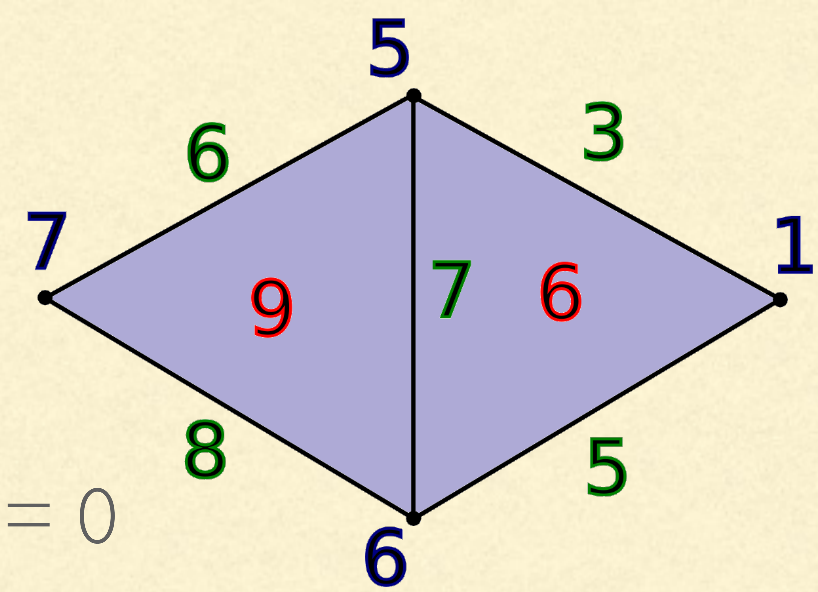


f : Σ → ℝ is called ***discrete Morse function*** if, for every simplex σ,

$$\# \{ \rho > \sigma \mid f(\rho) \leq f(\sigma) \} \leq 1$$

$$\# \{ \tau < \sigma \mid f(\tau) \geq f(\sigma) \} \leq 1$$

DISCRETE MORSE COMPLEX



A k-simplex σ is ***critical*** with index k if

$\# \{ \rho > \sigma \mid f(\rho) \leq f(\sigma) \} = \# \{ \tau < \sigma \mid f(\tau) \geq f(\sigma) \} = 0$

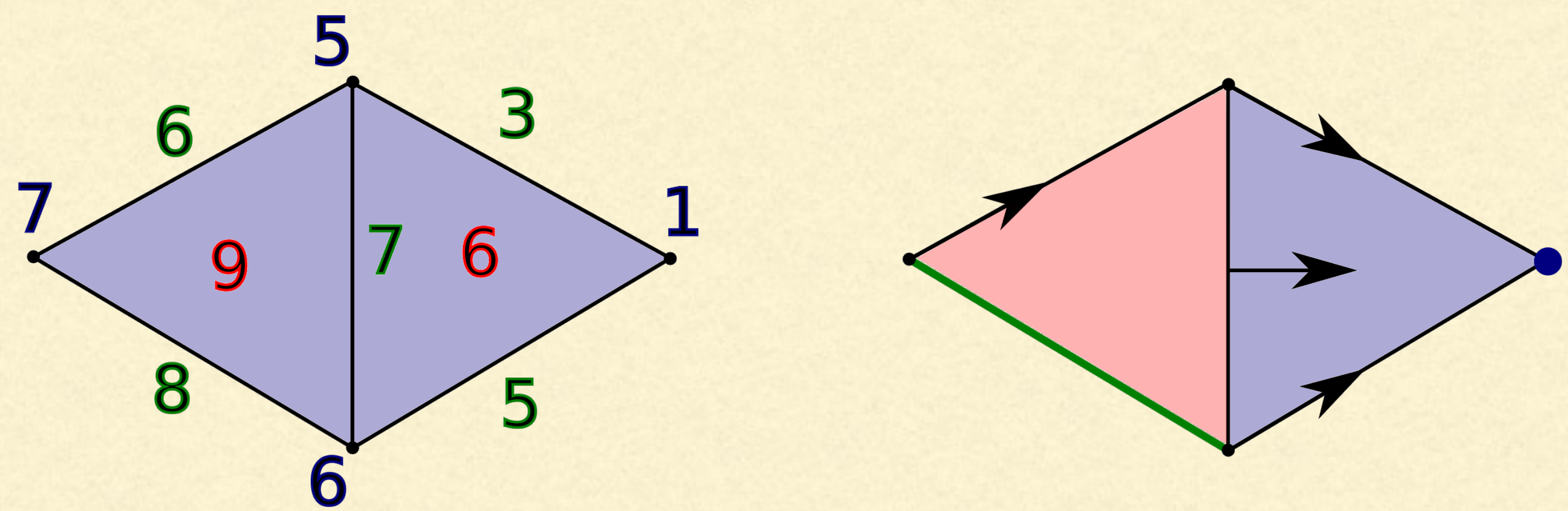
Critical simplices generate a chain complex $\mathscr{M}_*$ called

discrete Morse complex

Proposition. $H_k(\mathscr{M}_*) \cong H_k(\Sigma)$

DISCRETE MORSE FUNCTION AND GRADIENT VECTOR FIELD

A ***discrete vector field V*** on Σ is a collection of pairs of simplices $(\tau, \sigma) \in \Sigma \times \Sigma$ such that $\tau < \sigma$ and each simplex of Σ is in at most one pair of V



A discrete Morse function $f : \Sigma \to \mathbb{R}$ induces a discrete vector field on Σ

$$V = \{ (\tau, \sigma)\ \Sigma \times \Sigma \mid \tau < \sigma \text{ and } f(\tau) \geq f(\sigma) \}$$

called the ***gradient vector field*** of f

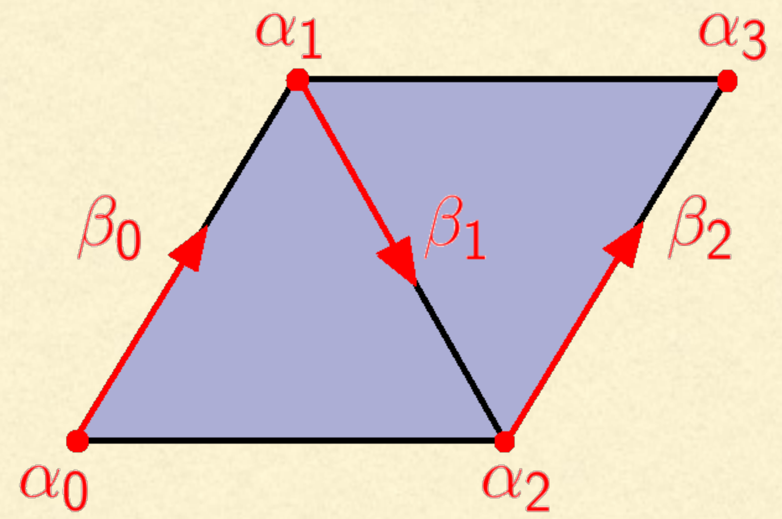
DISCRETE MORSE FUNCTION AND GRADIENT VECTOR FIELD

Given a discrete vector field V, a ***gradient path*** is a sequence of simplices of Σ

$$\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \dots, \alpha_{r-1}, \beta_{r-1}, \alpha_r$$

where $(\alpha_i, \beta_i) \in V$, $\alpha_{i+1} < \beta_i$ and $\alpha_i \neq \alpha_{i+1}$

A gradient path is a ***non-trivial closed path*** if $r \geq 0$ and $\alpha_0 = \alpha_r$



Theorem. A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed paths

REDUCTIONS AND COREDUCTIONS
[MROZEK ET AL. 2009]

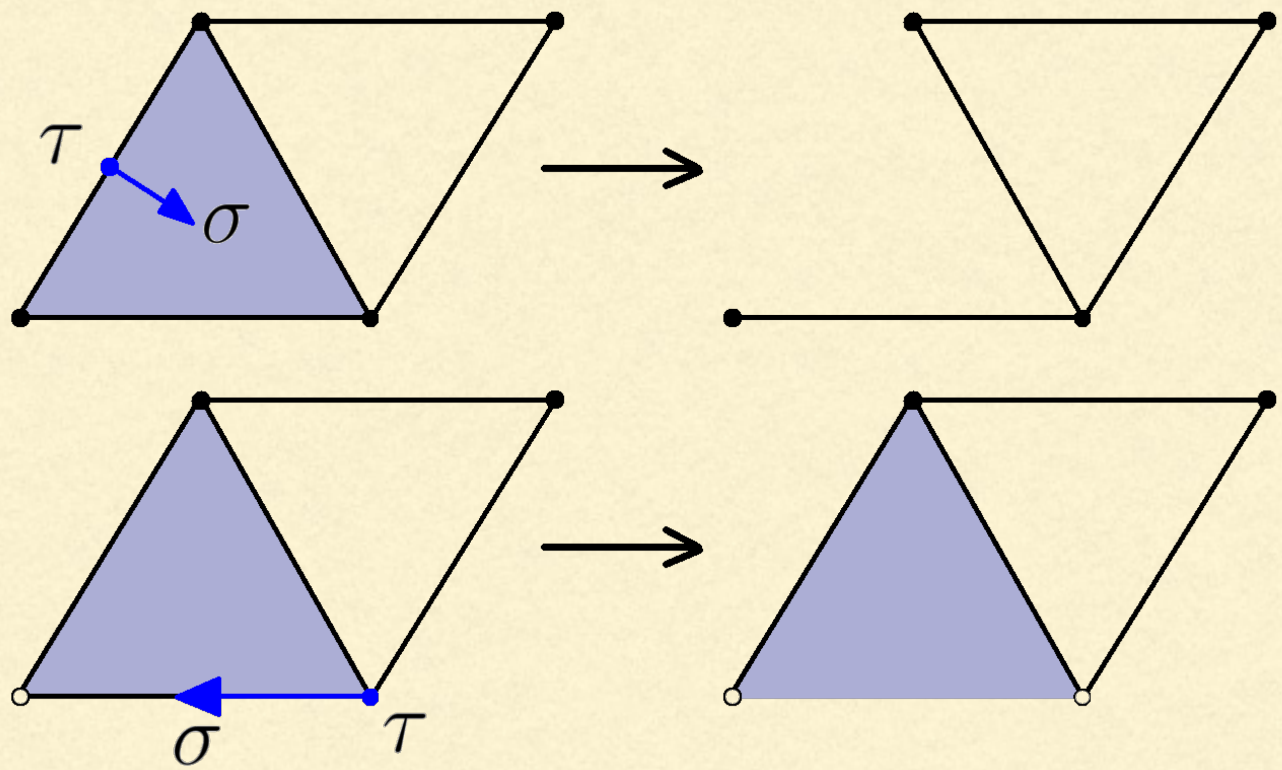
Let (τ, σ) be a pair of Σ such that $< \partial\sigma, \tau >= \pm 1$

(τ, σ) is a ***reduction pair*** if

$\mathrm{cbd}_\Sigma \tau = \{\sigma\}$

(τ, σ) is a ***coreduction pair*** if

$\mathrm{bd}_\Sigma\, \sigma = \{\tau\}$



Proposition. The removal of a reduction or of a coreduction pair is a homology-preserving operator

GRADIENT VECTOR FIELD BY REDUCTIONS
[BENEDETTI ET AL. 2009]

Input: Σ simplicial complex
Output: V gradient vector field, A set of critical simplices

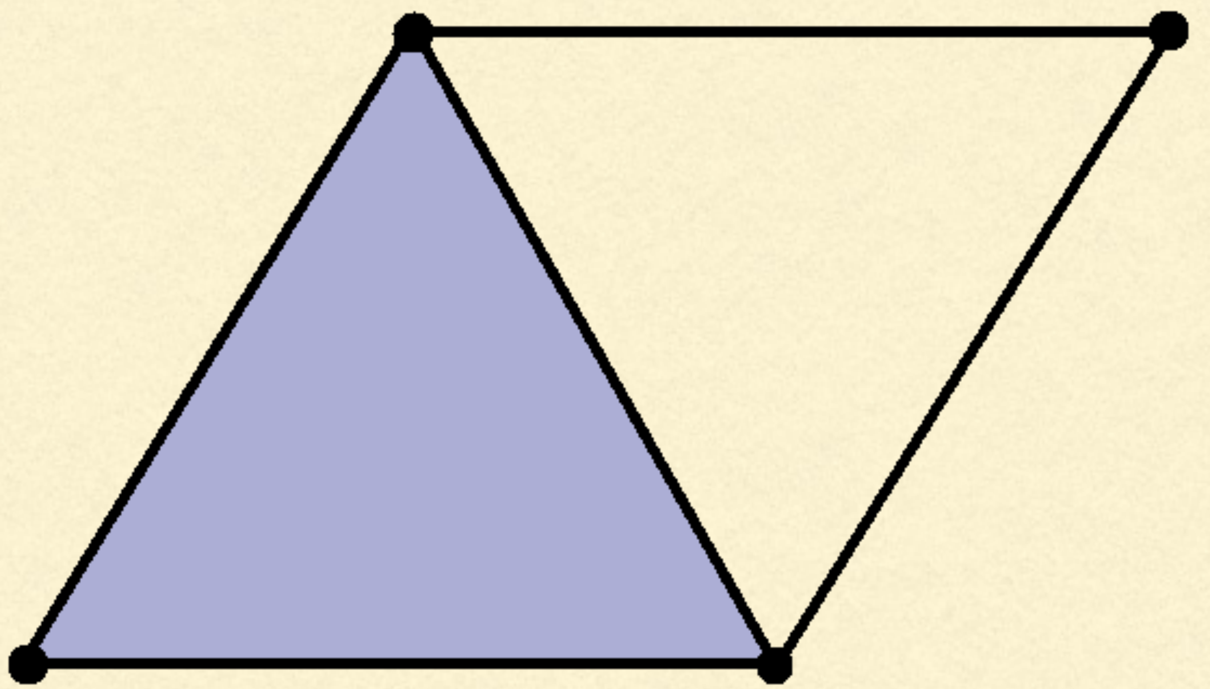
Set Σ' ← Σ, V ← ∅, A ← ∅

while Σ' ≠ ∅ do

  while Σ' admits a *reduction pair* (τ, σ) do
    V ← V ∪ { (τ, σ) }
    Σ' ← Σ' \ { τ, σ }
  end while

  Let η be a *top simplex* in Σ'
    A ← A ∪ { η }
    Σ' ← Σ' \ { η }

end while



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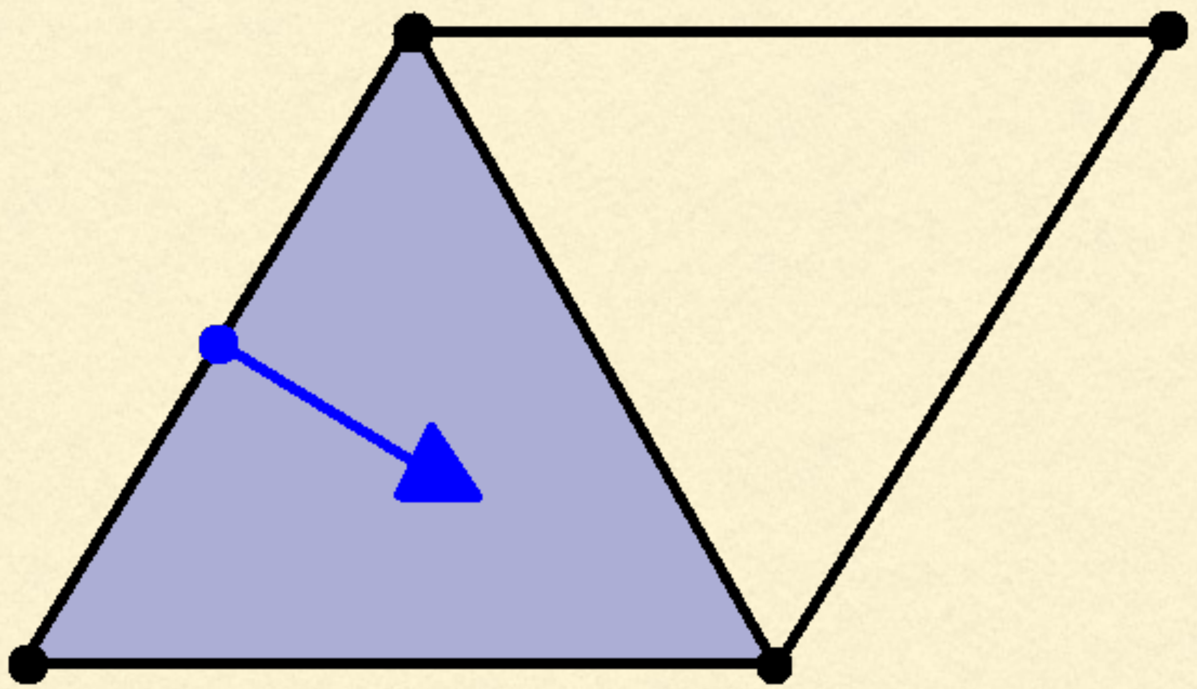

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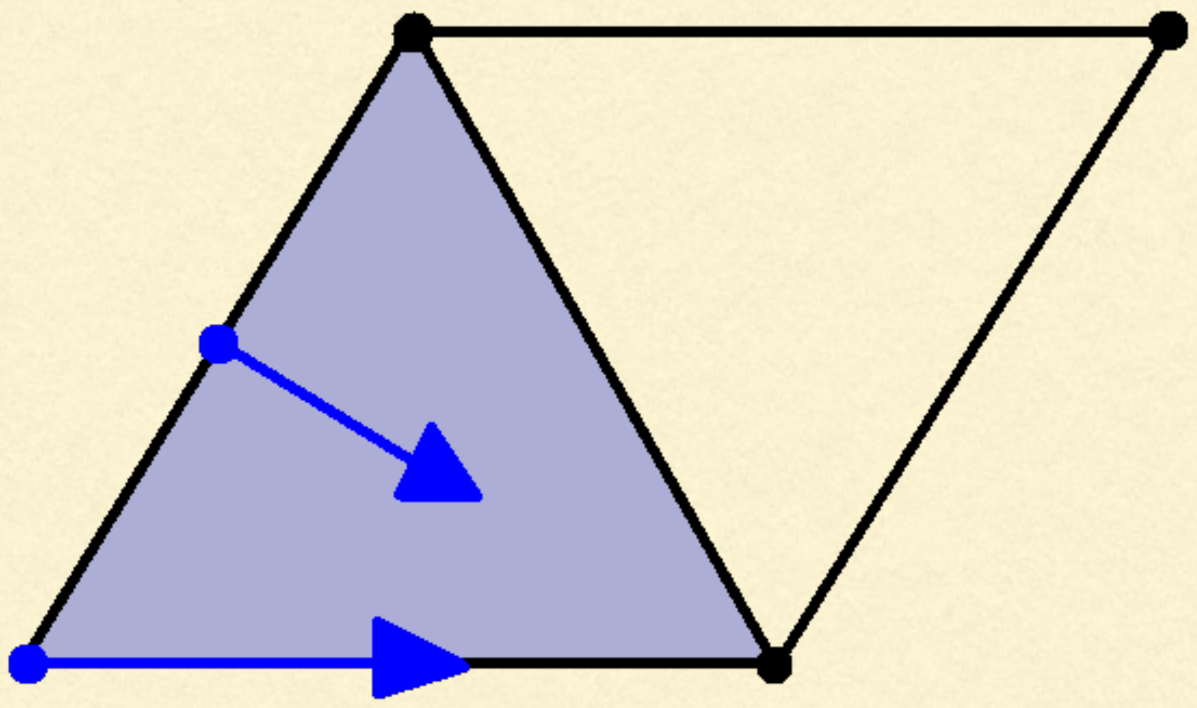

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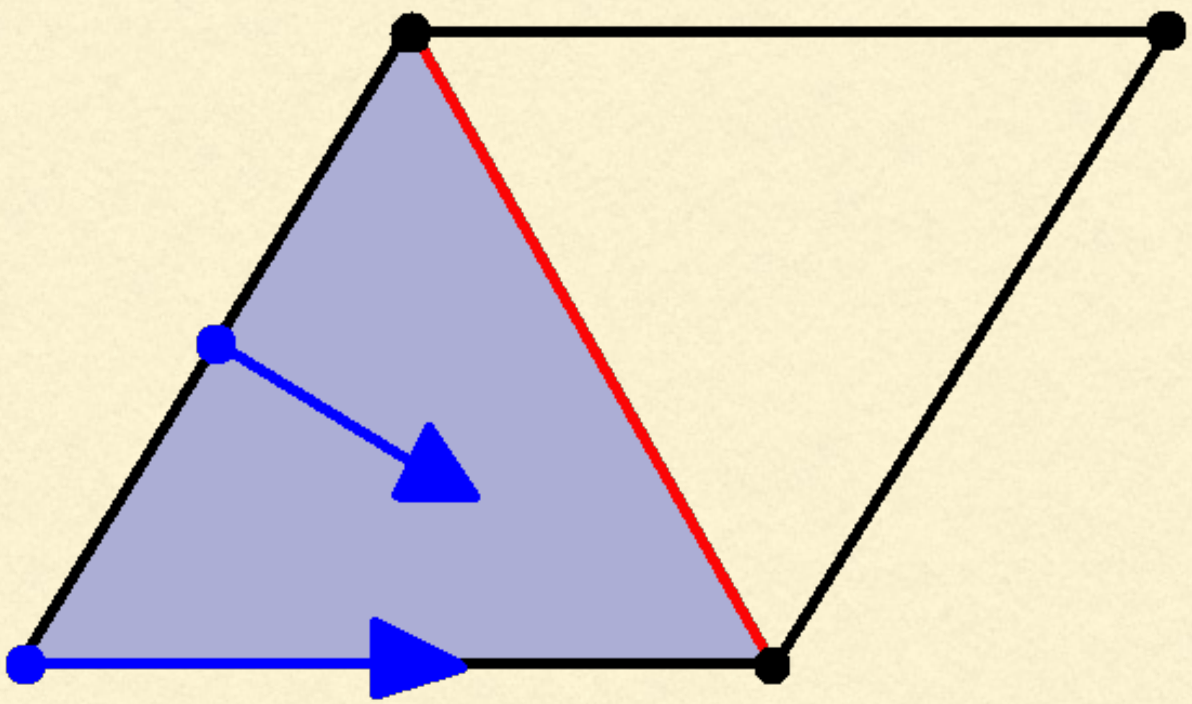

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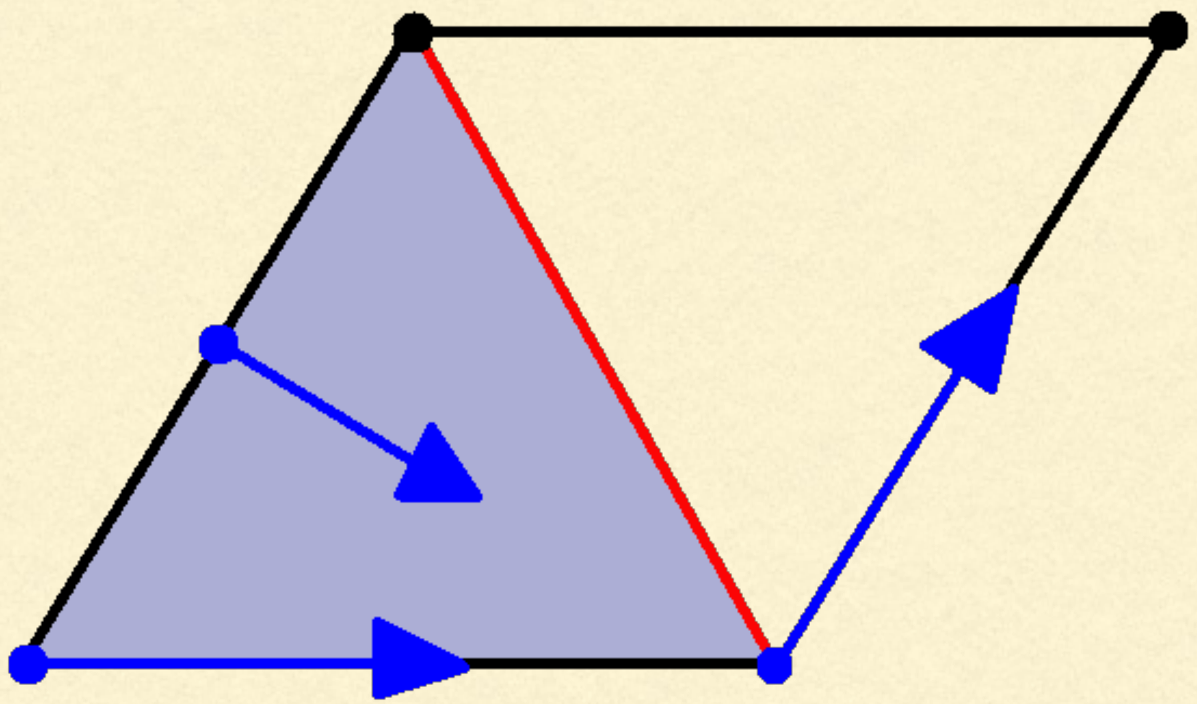

GRADIENT VECTOR FIELD BY REDUCTIONS
[BENEDETTI ET AL. 2009]

Input: Σ simplicial complex
Output: V gradient vector field, A set of critical simplices

Set $\Sigma' \leftarrow \Sigma$, $V \leftarrow \varnothing$, $A \leftarrow \varnothing$

while $\Sigma' \neq \varnothing$ do

  while Σ' admits a *reduction pair* (τ, σ) do
    $V \leftarrow V \cup \{ (\tau, \sigma) \}$
    $\Sigma' \leftarrow \Sigma' \setminus \{ \tau, \sigma \}$
  end while

  Let η be a *top simplex* in Σ'
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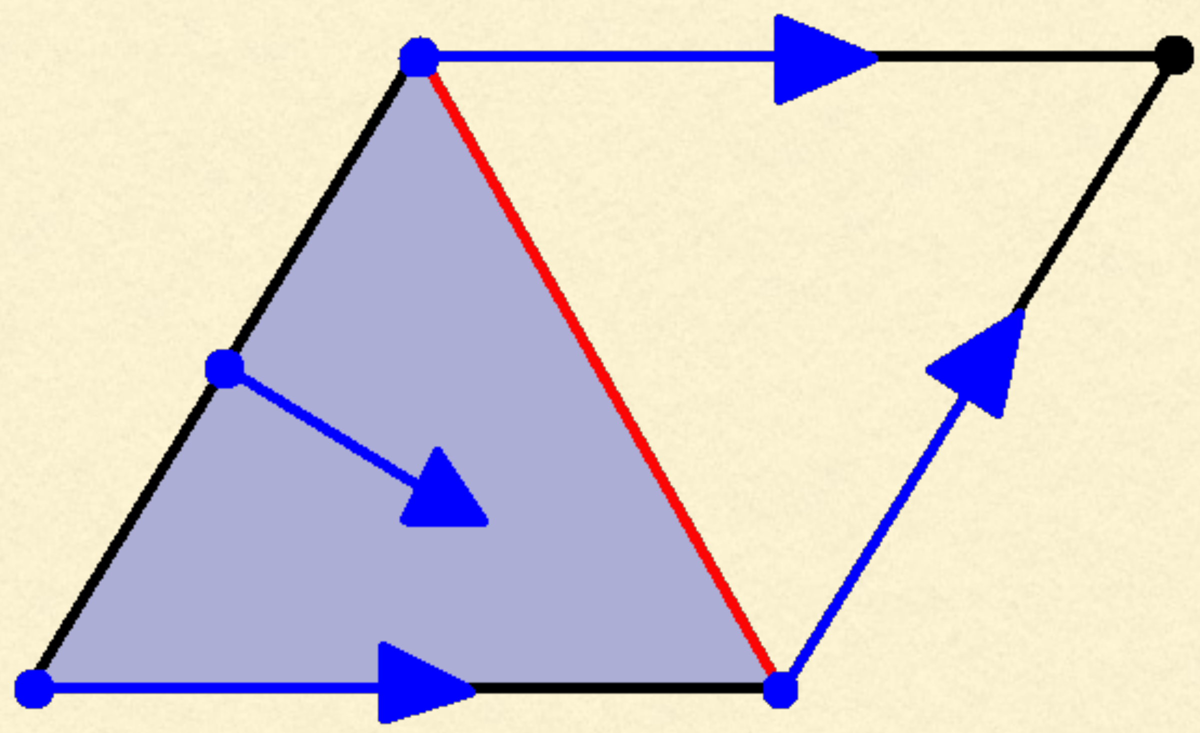

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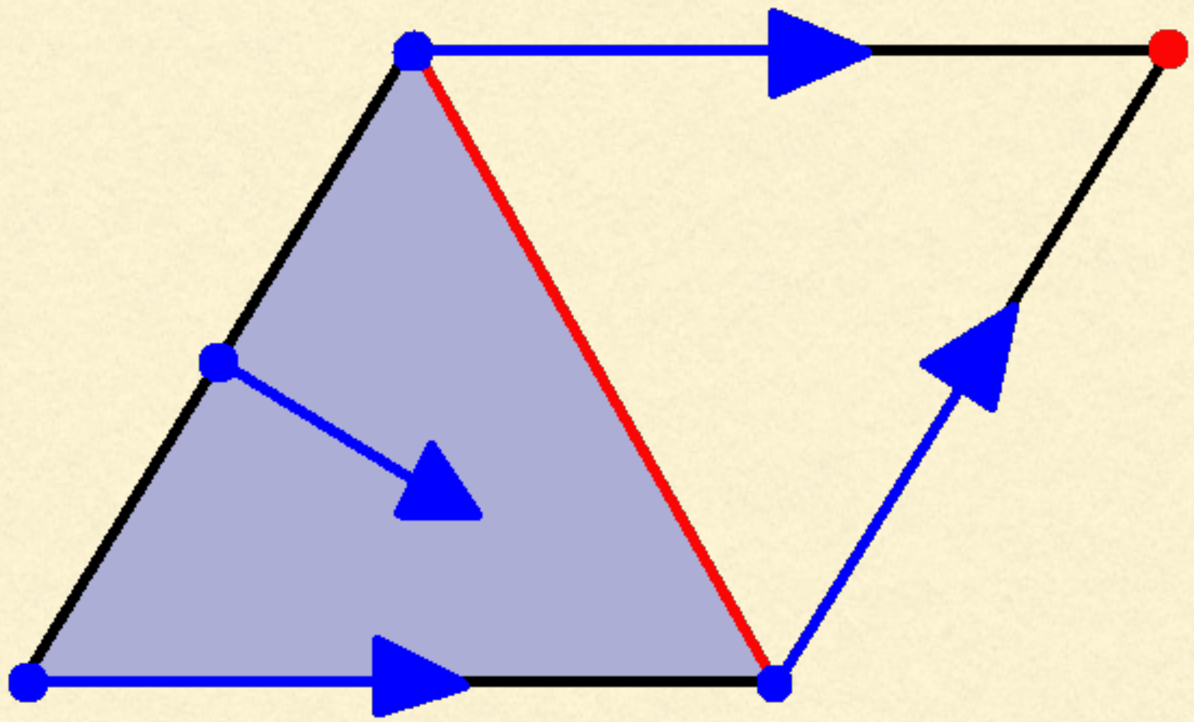

GRADIENT VECTOR FIELD BY COREDUCTIONS [HARKER ET AL. 2010]

Input: Σ simplicial complex
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Set Σ' ← Σ, V ← ∅, A ← ∅

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Let η be a *free simplex* in Σ'
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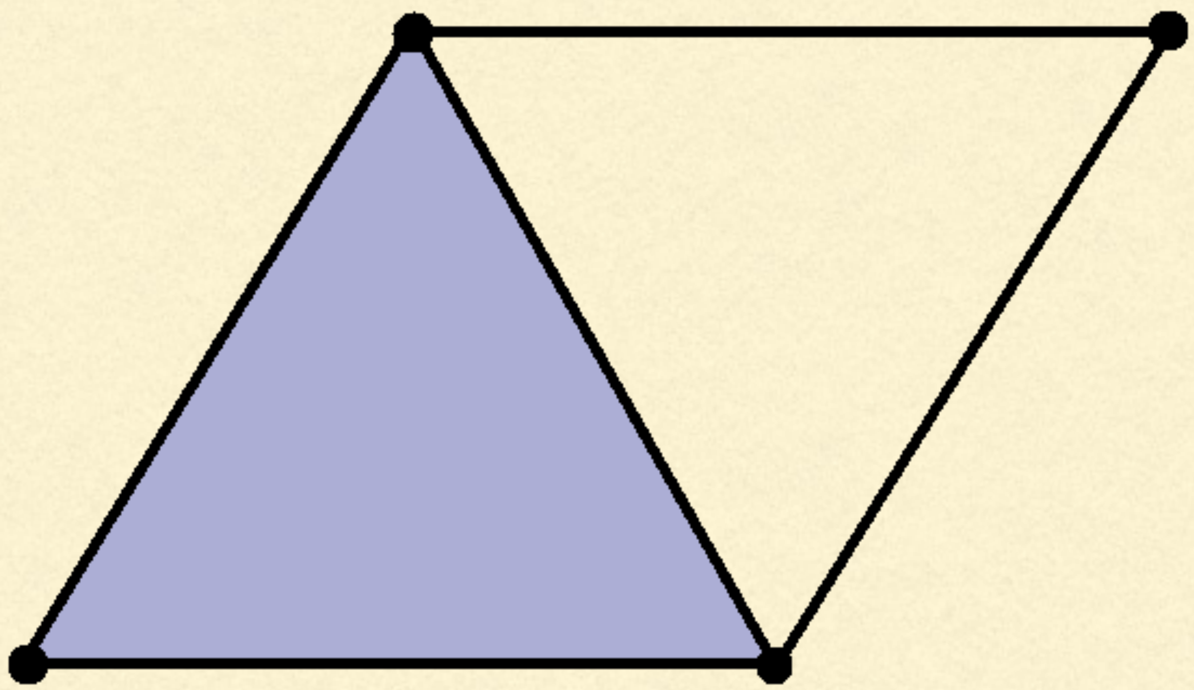

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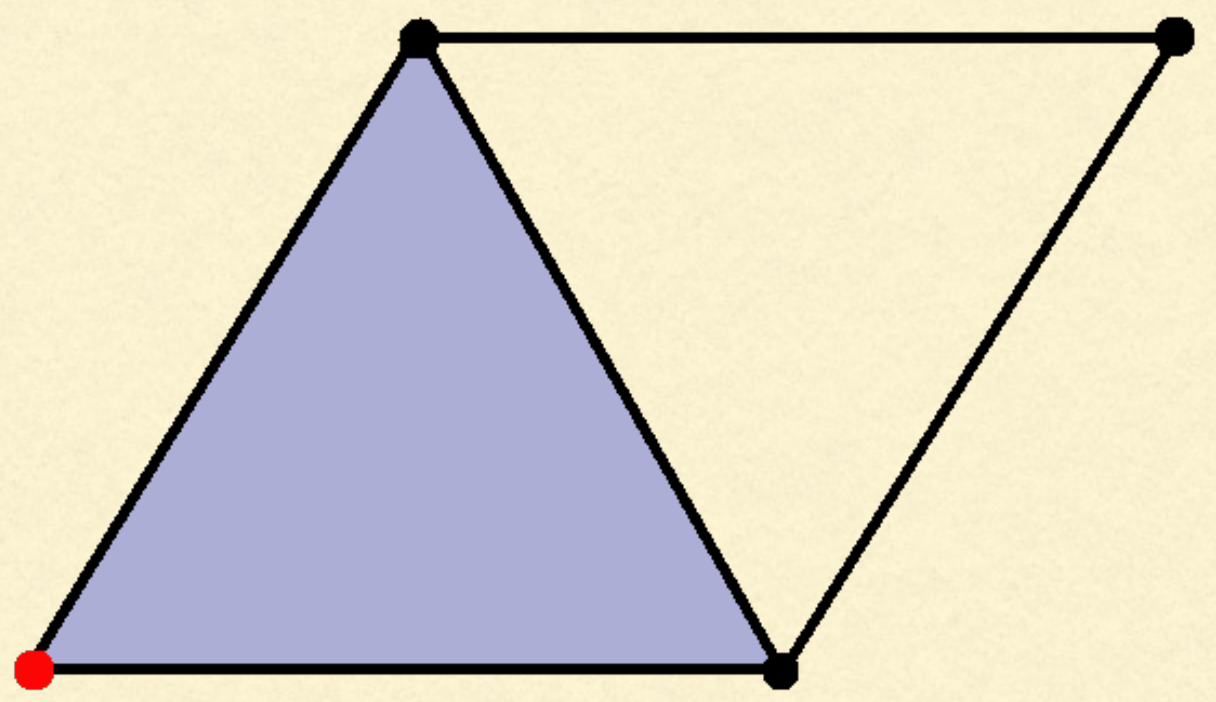

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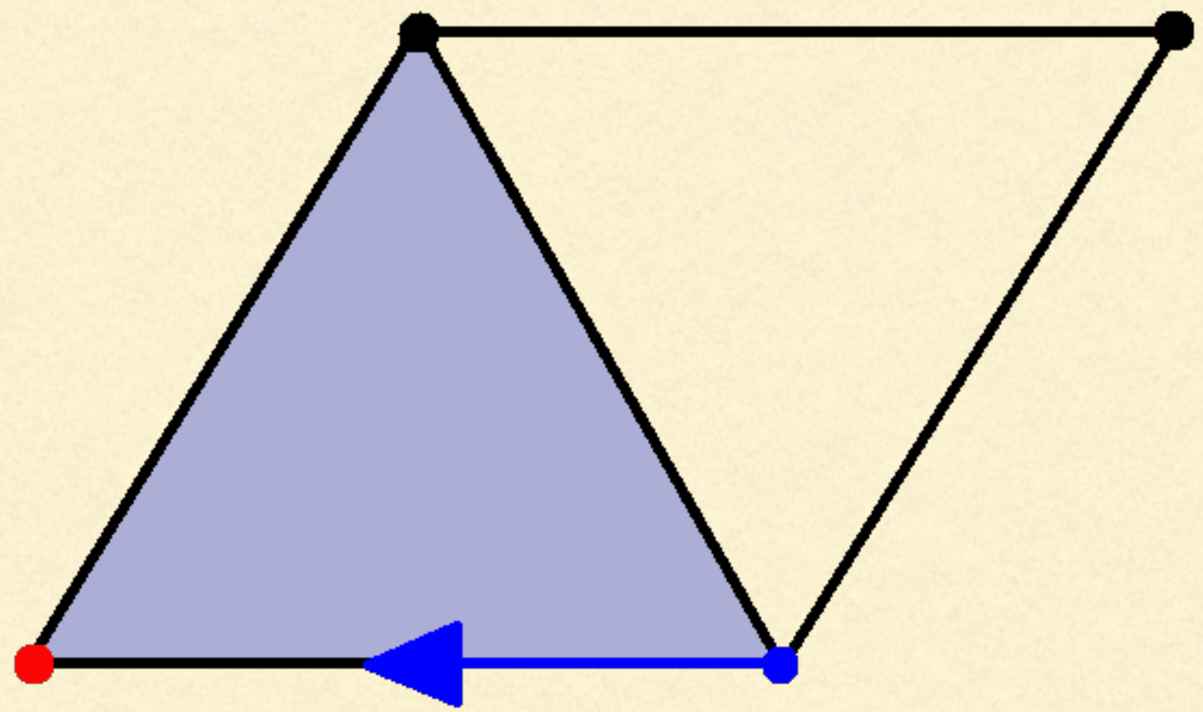

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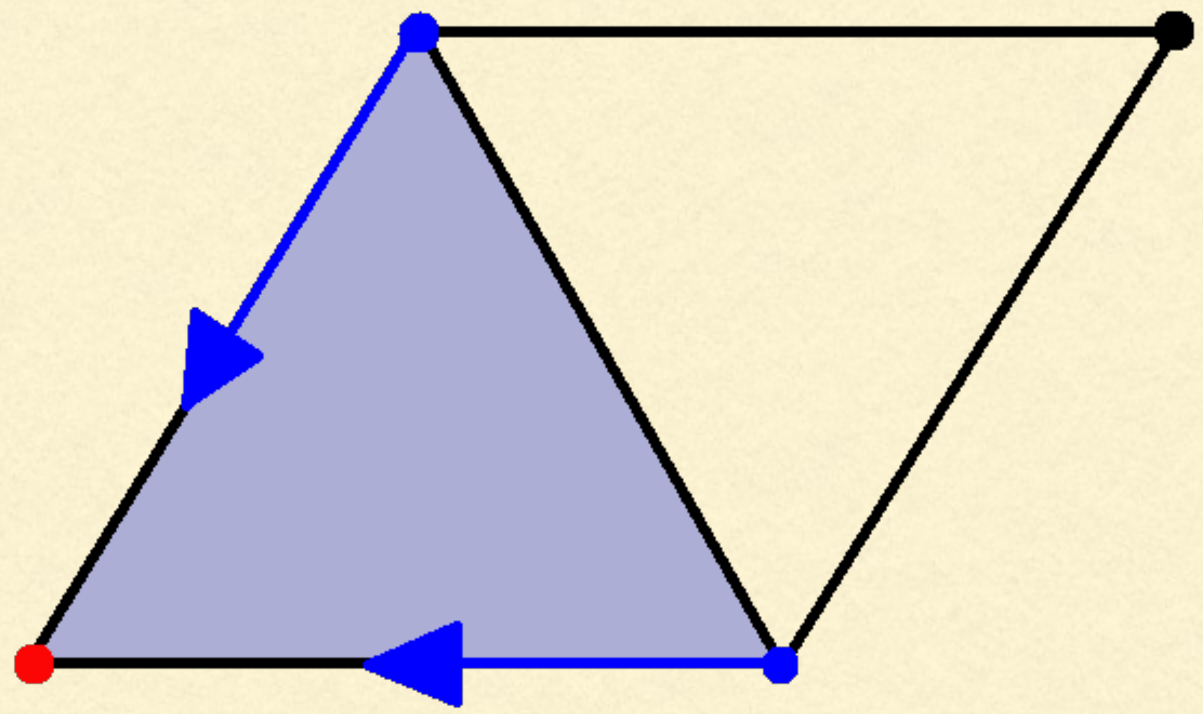

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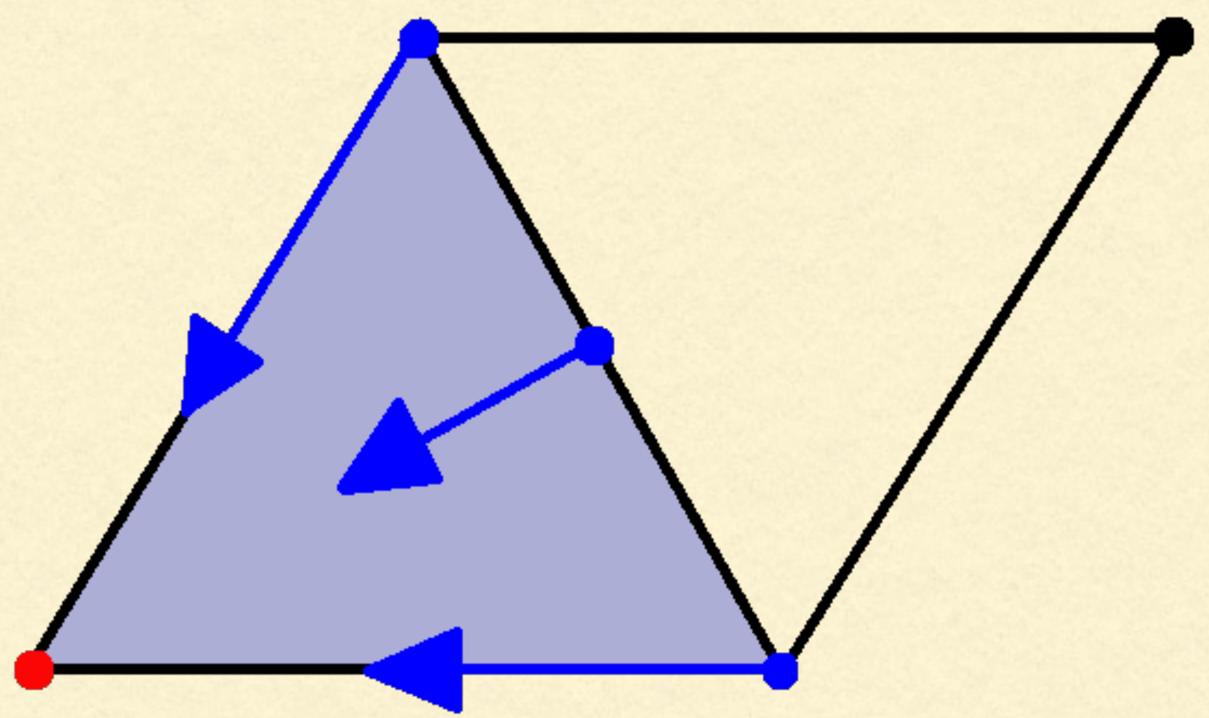

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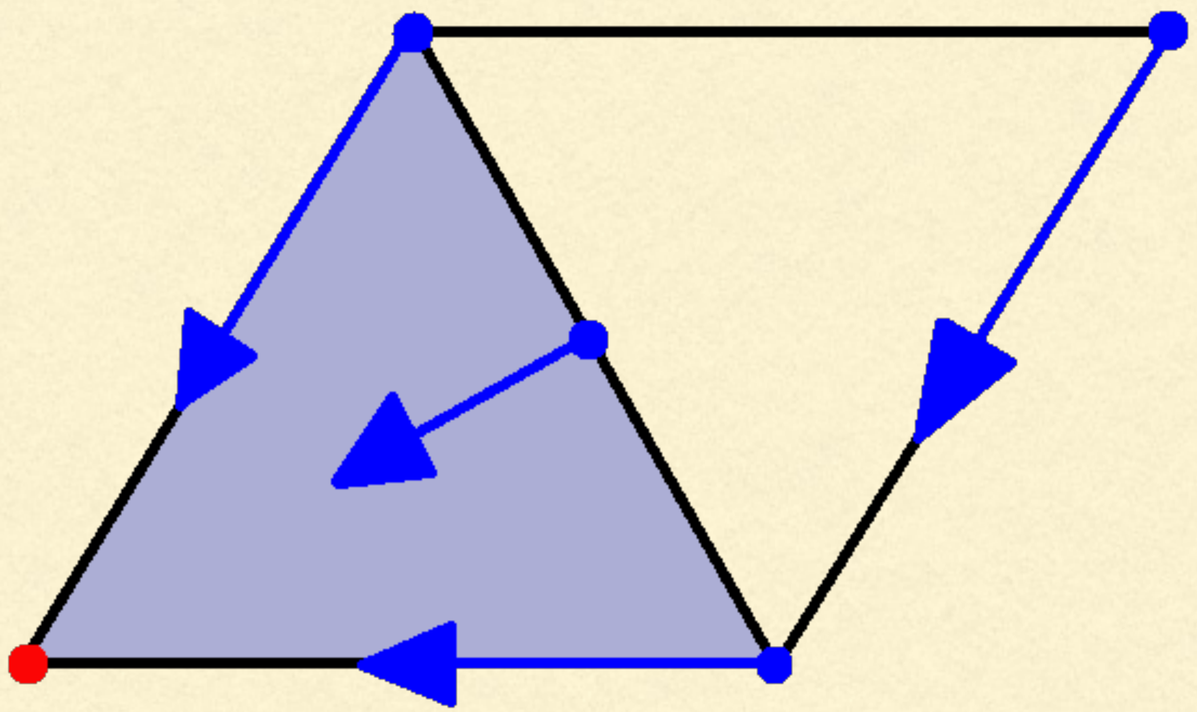

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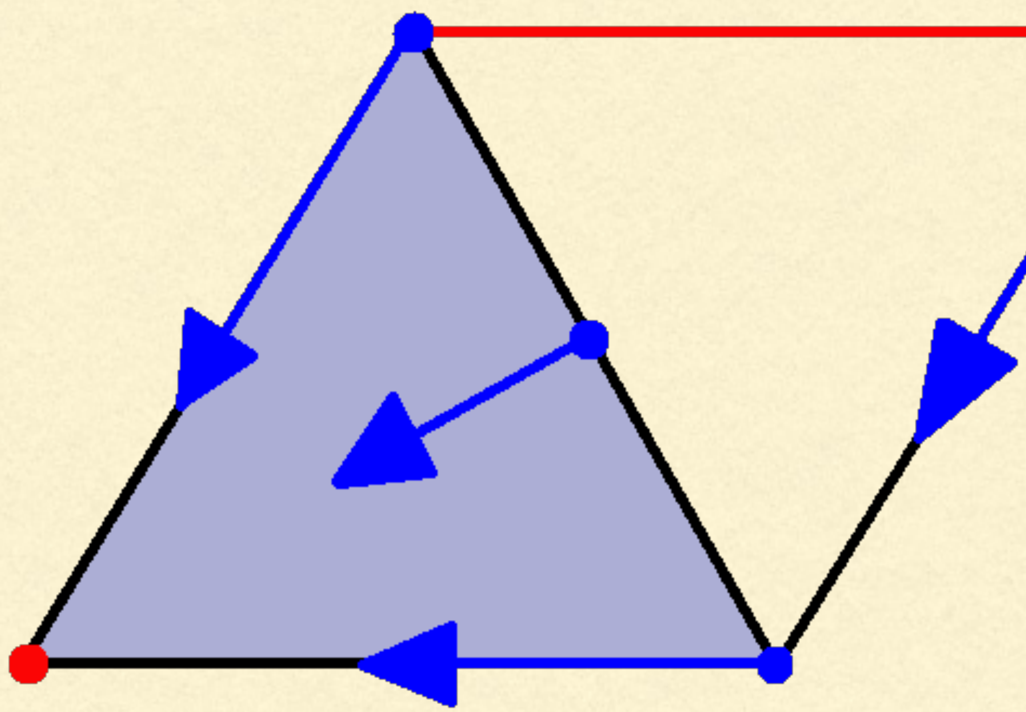


REDUCTION-BASED AND COREDUCTION-BASED APPROACHES

Proposition.
Both the algorithms produce a gradient vector field on Σ

The same holds even if the following condition is not satisfied

(☆) A new critical simplex is created only if no more [co]reduction pair is available

EQUIVALENCE OF REDUCTION-BASED AND COREDUCTION-BASED APPROACH

Which approach is able to compute a gradient vector field with less critical simplices?

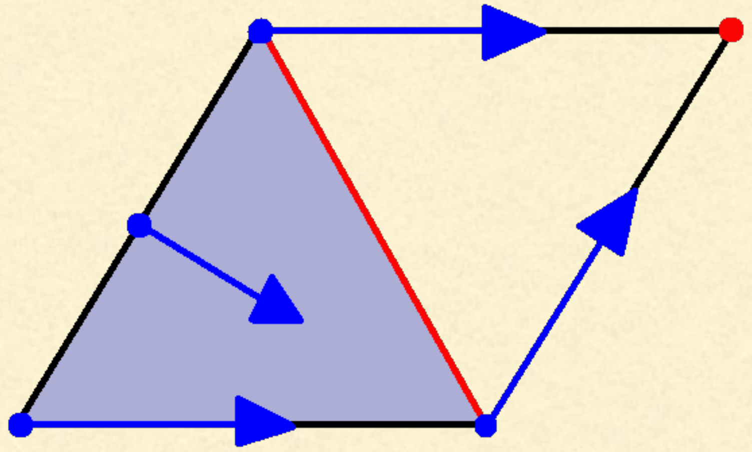
Reduction-based and coreduction-based approaches are ***equivalent***

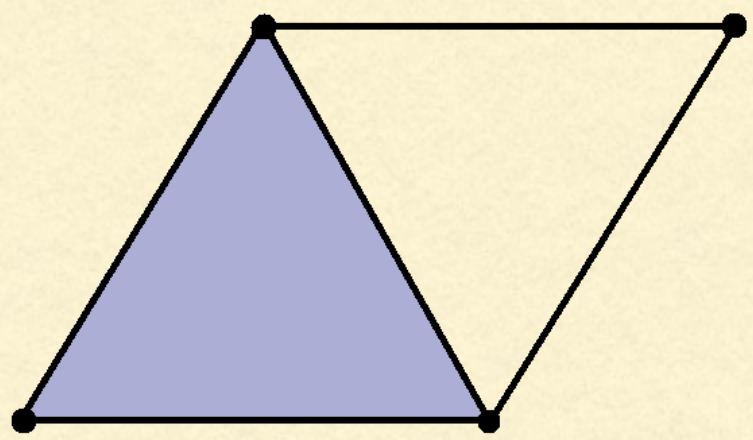
Proposition. Given a simplicial complex Σ and the gradient vector field V produced by a reduction-based algorithm, it is always possible to obtain the same gradient vector field with a coreduction-based algorithm
The reverse is also true

EQUIVALENCE OF REDUCTION-BASED AND COREDUCTION-BASED APPROACH

Proof's guidelines:

- Consider a simplicial complex Σ and run the reduction-based approach on it
- Take the sequence of reduction pairs and top simplex removals operated by the algorithm
- Reverse the order of the sequence: this new sequence represents for Σ a performable sequence of coreduction pairs and free simplex removals





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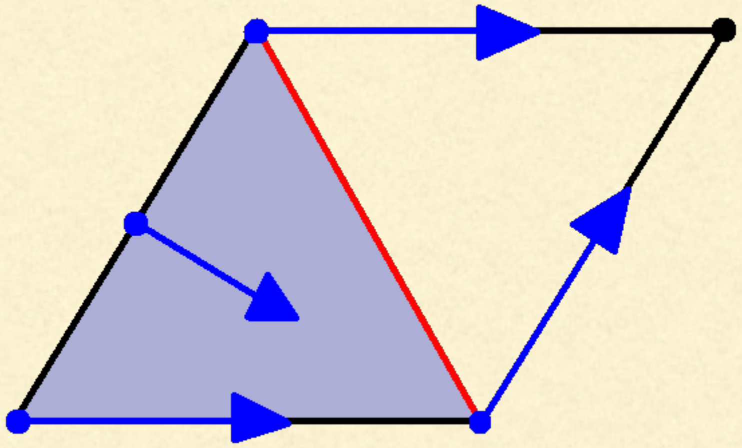

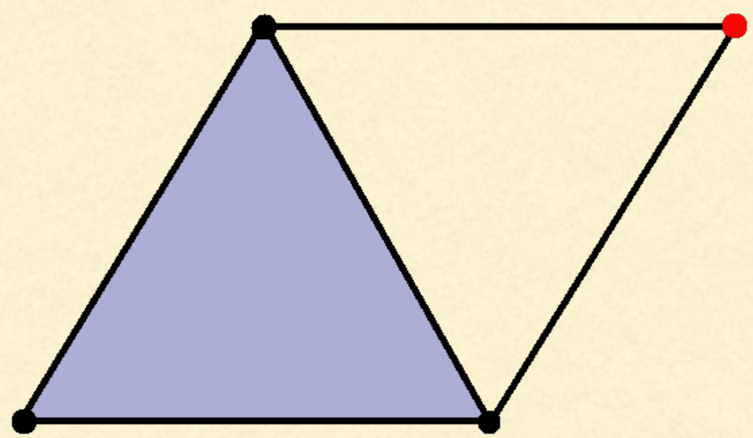

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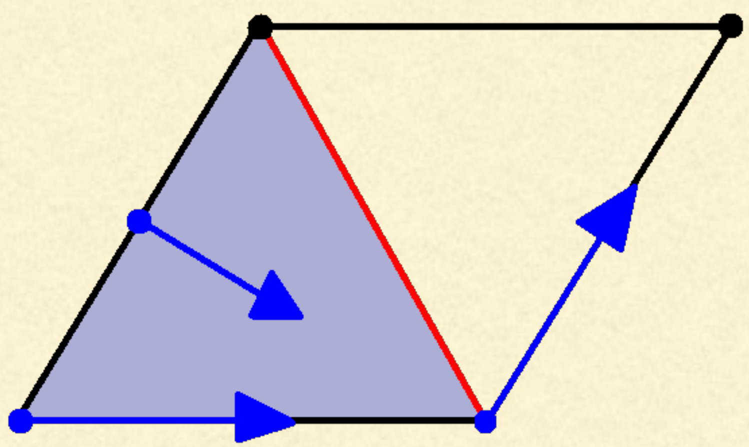

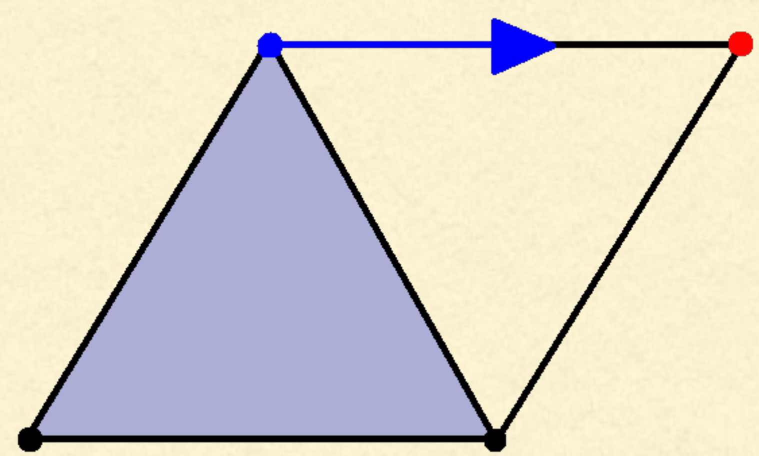

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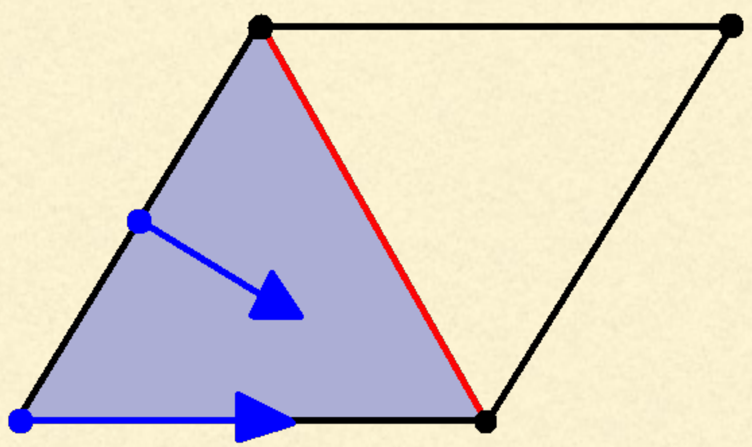

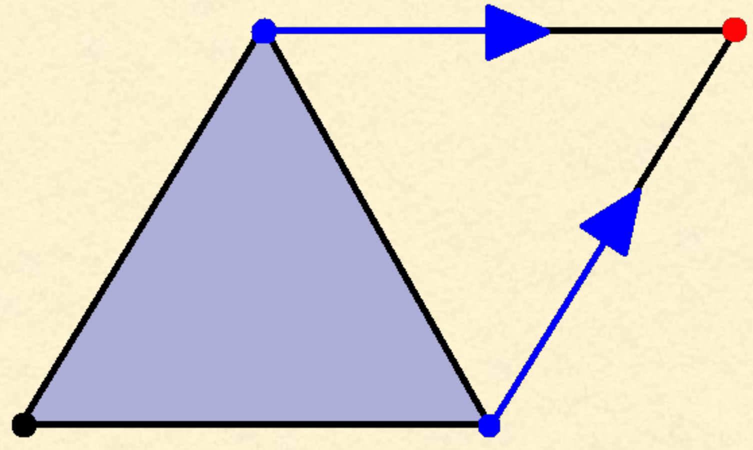

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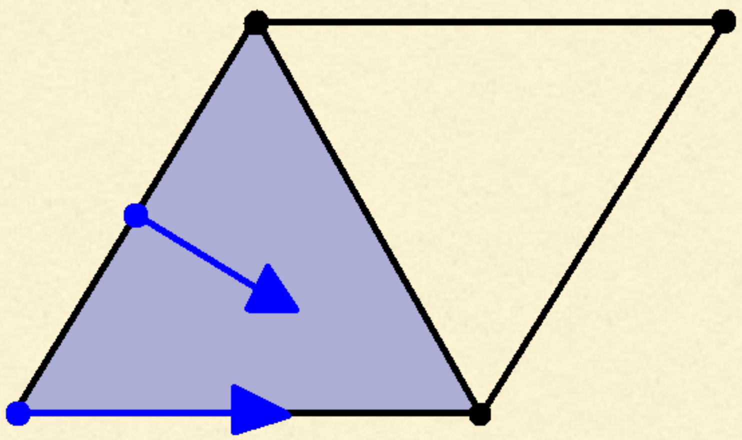

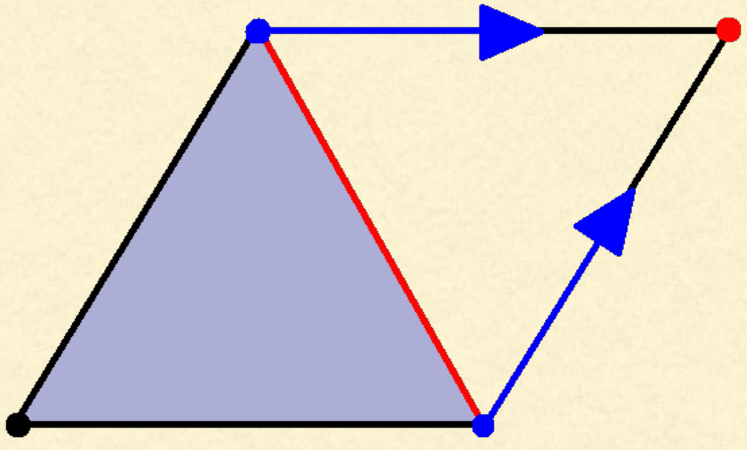

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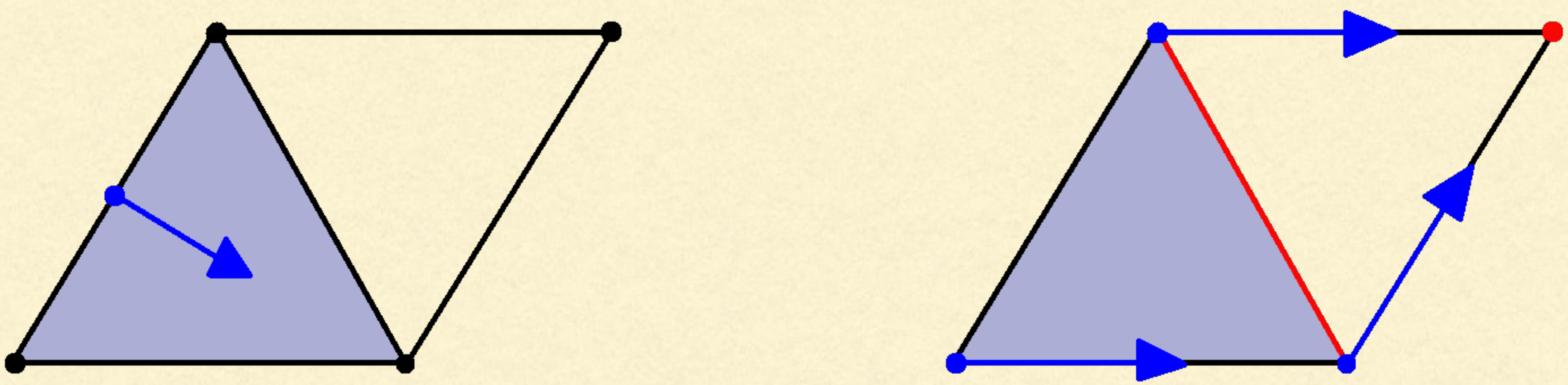

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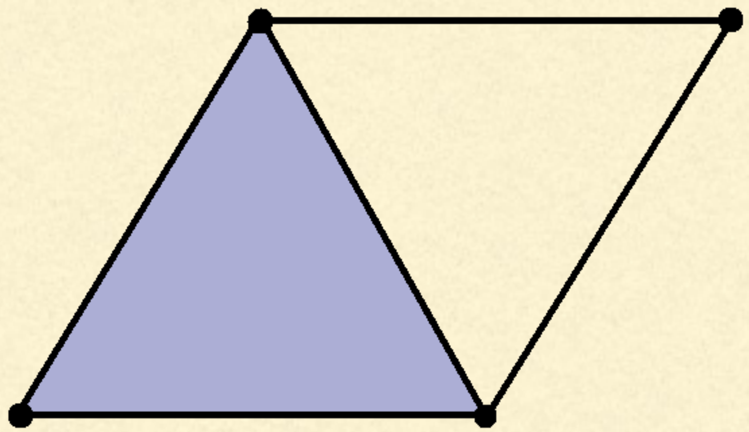
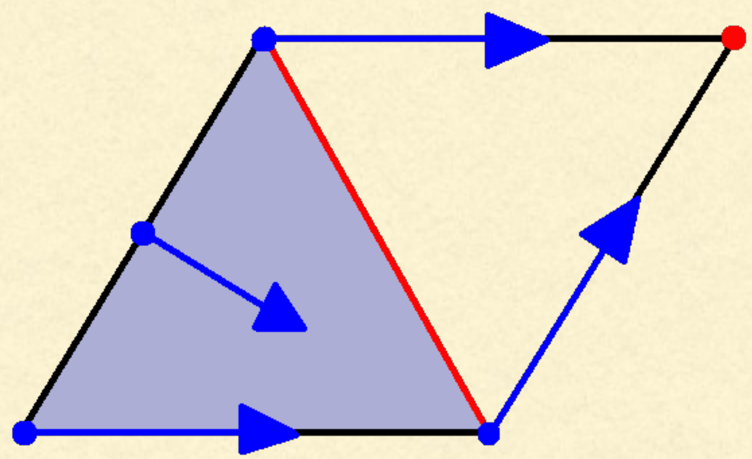




EQUIVALENCE OF REDUCTION-BASED AND COREDUCTION-BASED APPROACH

Remark. It is not true, in general, that, given a gradient vector field V built by using a reduction-based algorithm satisfying condition (☆), V can be produced by a coreduction-based algorithm for which (☆) holds

INTERLEAVING REDUCTIONS AND COREDUCTIONS

Another class of approaches interleaving reductions and coreductions could be proposed

Proposition. Given a simplicial complex Σ, any discrete vector field produced by an algorithm performing reduction and coreduction pairs, removals of top and of free simplices is a gradient vector field

Each interleaved approach has equivalent capabilities

Proposition. Given a simplicial complex Σ and the gradient vector field V produced by an interleaved algorithm, it is always possible to obtain the same gradient vector field with a reduction-based algorithm or, equivalently, with a coreduction-based algorithm

OUR ALGORITHM

We developed a new ***coreduction-based algorithm*** for ***simplicial homology computation*** based on a highly ***efficient data structure*** and on a ***compact encoding of the gradient vector field***

Main features:

- Compact encoding of the simplicial complex
- Computation of a gradient vector field through a coreduction-based approach
- Compact encoding of the gradient vector field on the data structure
- Extraction of boundary maps
- Retrieval of simplicial homology and homology generators

COMPACT ENCODING OF THE SIMPLICIAL COMPLEX

A simplicial complex Σ is encoded by ***Generalized Indexed data structure with Adjacencies (IA*)*** [Canino et al. 2011]

IA* encodes:

- top simplices and vertices of Σ
- partial (co)boundary and adjacency relations for each encoded simplex

Incidence Graph (IG) encodes:

- all the simplices of Σ
- (immediate) boundary and coboundary for each simplex

The IA* data structure is generally more compact than IG:

- about 46% in dimension 2
- about 70% in dimension 3
- up to 170 times for special cases in dimension 8

COMPACT ENCODING OF THE GRADIENT VECTOR FIELD

Encoding of the gradient vector field is associated ***only with the top simplices***

Each top k-simplex σ encodes a ***bitvector*** of length $\sum_{i=1}^{k} \binom{k+1}{i+1}(i+1)$ representing ***all the possible pairings*** on its boundary.

For efficiency, an additional bitvector, denoted as paired(σ), is encoded for each top simplex:

paired(σ) encodes, for each simplex τ in the boundary of σ, whether τ is paired

COMPACT ENCODING OF THE GRADIENT VECTOR FIELD

Let η, τ be two paired simplices both in σ of dimension j and j+1 respectively; the resulting pair will be encoded in the following position of the bitvector of σ

$$\textit{start} + (j+2)\ \textit{pos}\tau + \textit{pos}\eta$$

where

- start $= \sum_{i=j+2}^{k} \binom{k+1}{i+1}(i+1)$
- posτ is the position of τ on the boundary of σ
- posη is the position of η on the boundary of τ

EXPERIMENTAL EVALUATION

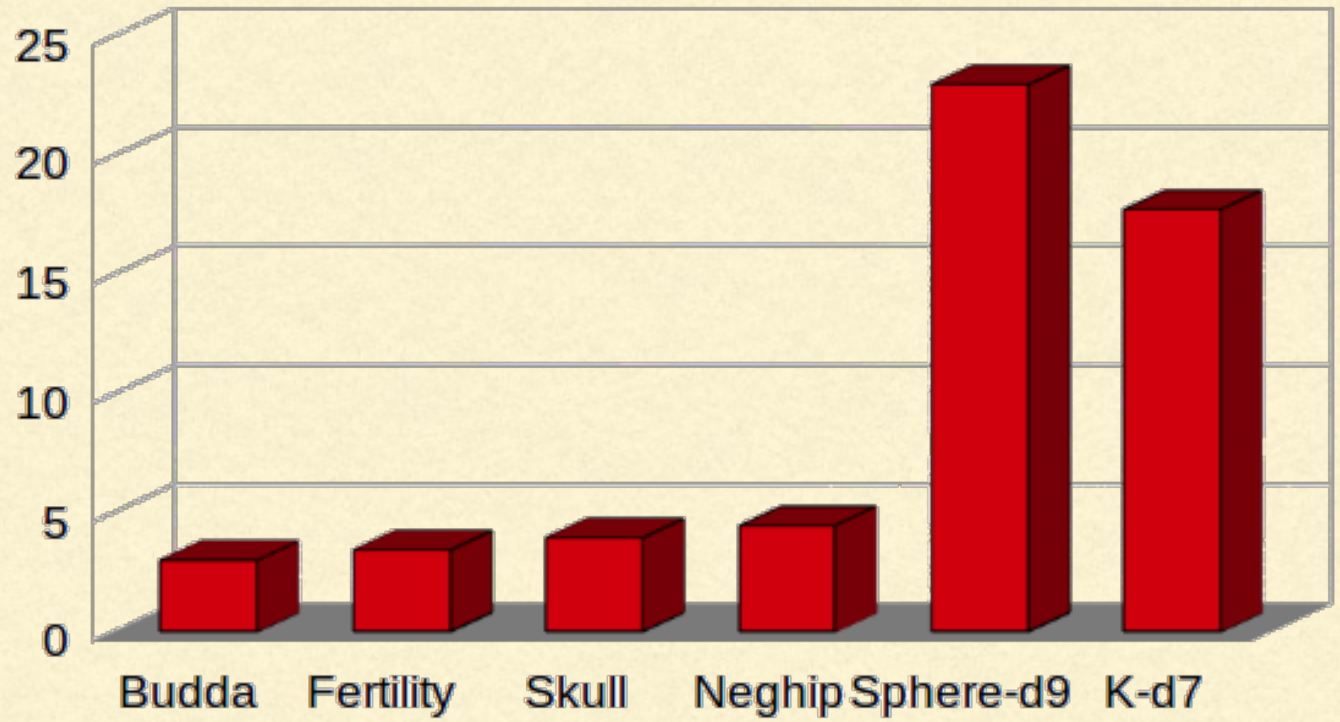
Dataset	d	n	n_0	n_{top}
Buddha	2	3.2M	0.54M	1.08M
Elephant	2	9.2M	1.5M	3.07M
Fertility	2	1.4M	0.24M	0.48M
Skull	3	0.75M	37K	0.15M
Neghip	3	2.1M	93K	0.48M
7Klein	7	0.1M	0.11K	0.6K
9Sphere	9	0.22M	2.0K	911

Column d indicates the complex dimension, n indicates the total number of simplexes, n_0 the number of vertices and n_{top} the number of top simplices

EXPERIMENTAL EVALUATION

We compared our performances with Perseus [Nanda 2012]

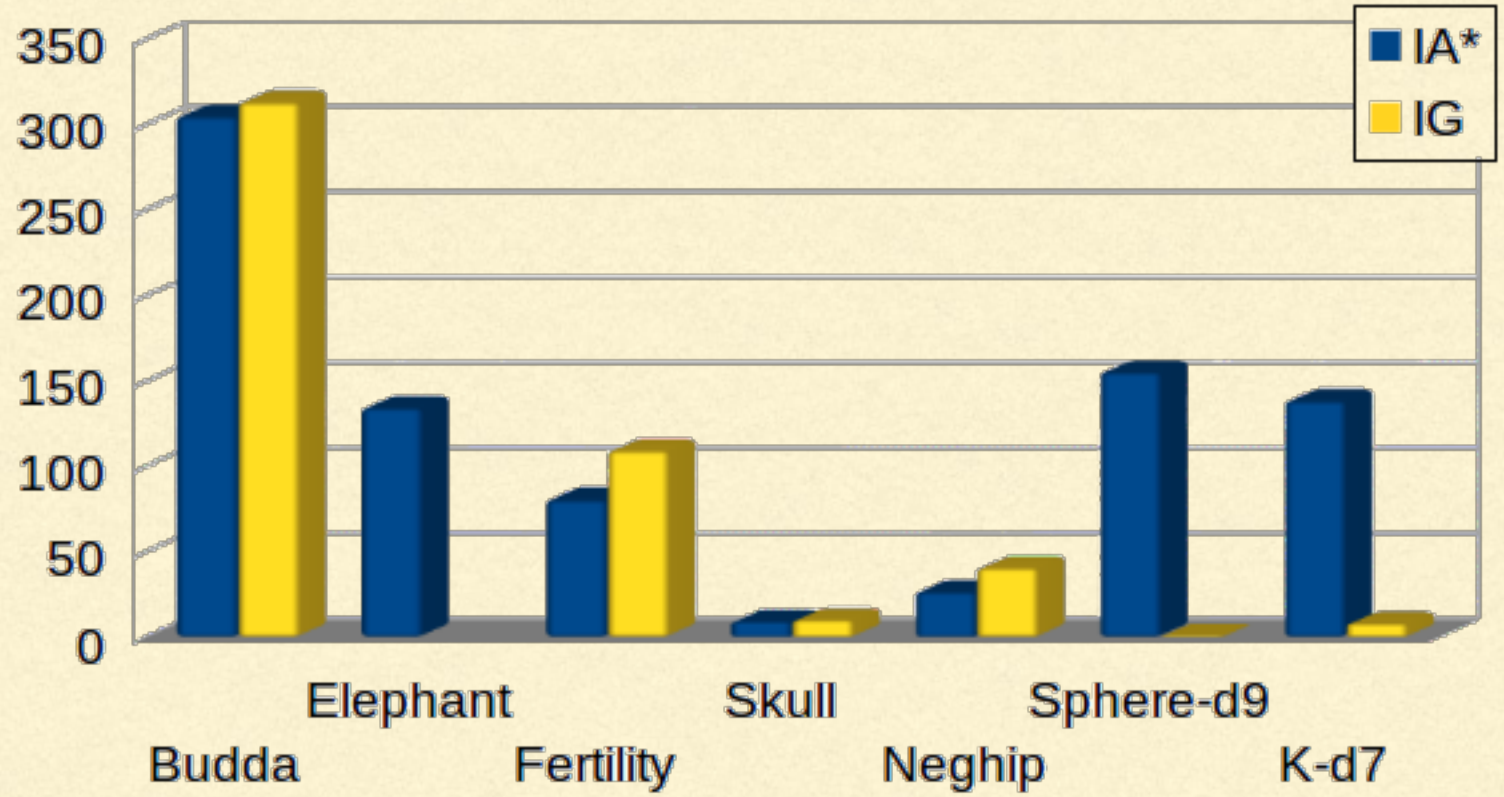
IA vs IG: storage cost*



Ratio (IG/IA*) of the storage costs of IG and IA* data structures endowed with gradient vector field encoding

EXPERIMENTAL EVALUATION

IA vs IG: timings*



Comparisons of timings for the homology computation algorithms based on the IA* and IG data structures

CONCLUSIONS

We have

- proven the ***equivalence of different methods*** based on homology-preserving operators to compute a gradient vector field
- developed a new ***algorithm based on coreductions***, on a ***space-efficient representation*** of the simplicial complex and on a ***compact encoding*** of the gradient vector field

We plan to develop

- an efficient encoding based for a simplicial complex in arbitrary dimension on the ***stellar tree*** data structure [Weiss et al. 2011]
- a new algorithm for ***persistent homology*** computation