“Persistent Homology” Summer School - Rabat

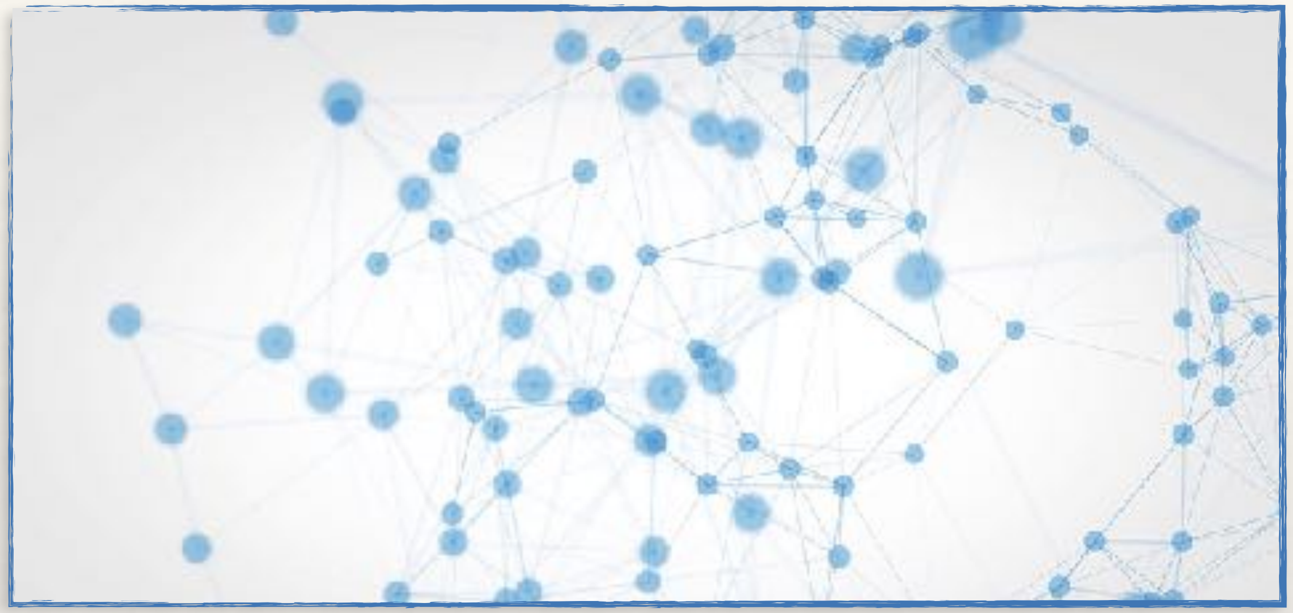
Persistent Homology in Complex Network Analysis

Ulderico Fugacci

Kaiserslautern University of Technology
Department of Computer Science

July 7, 2017

Anything has Shape



"Data has shape and
shape has meaning"

Gunnar Carlsson

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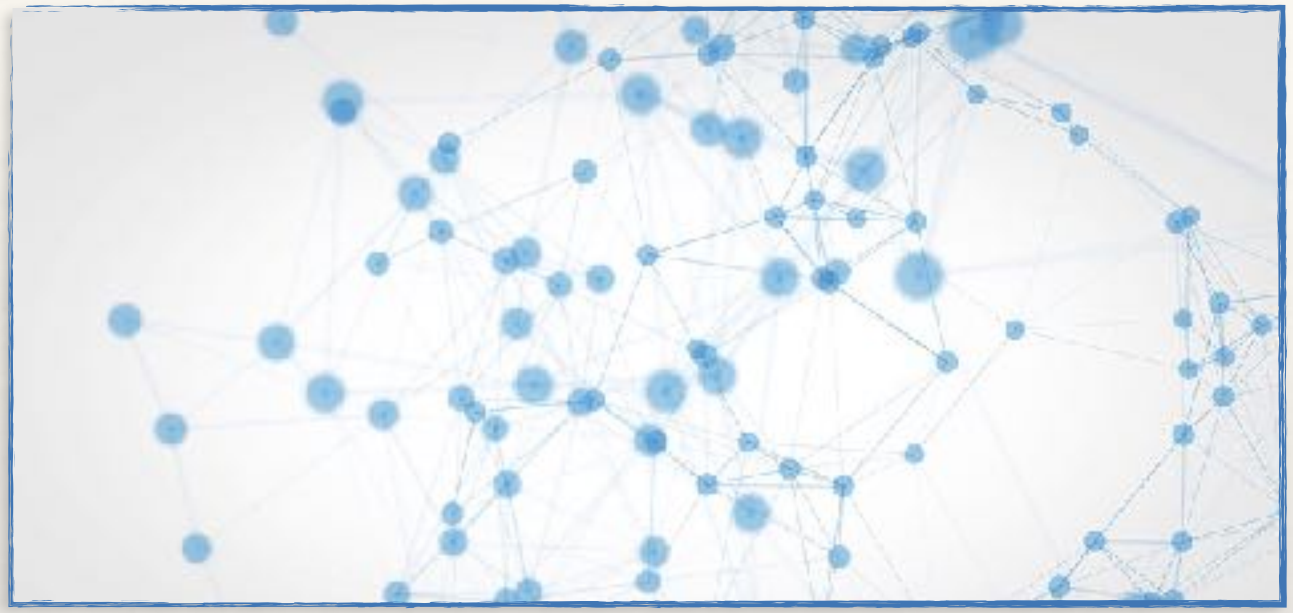

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Persistent Homology:

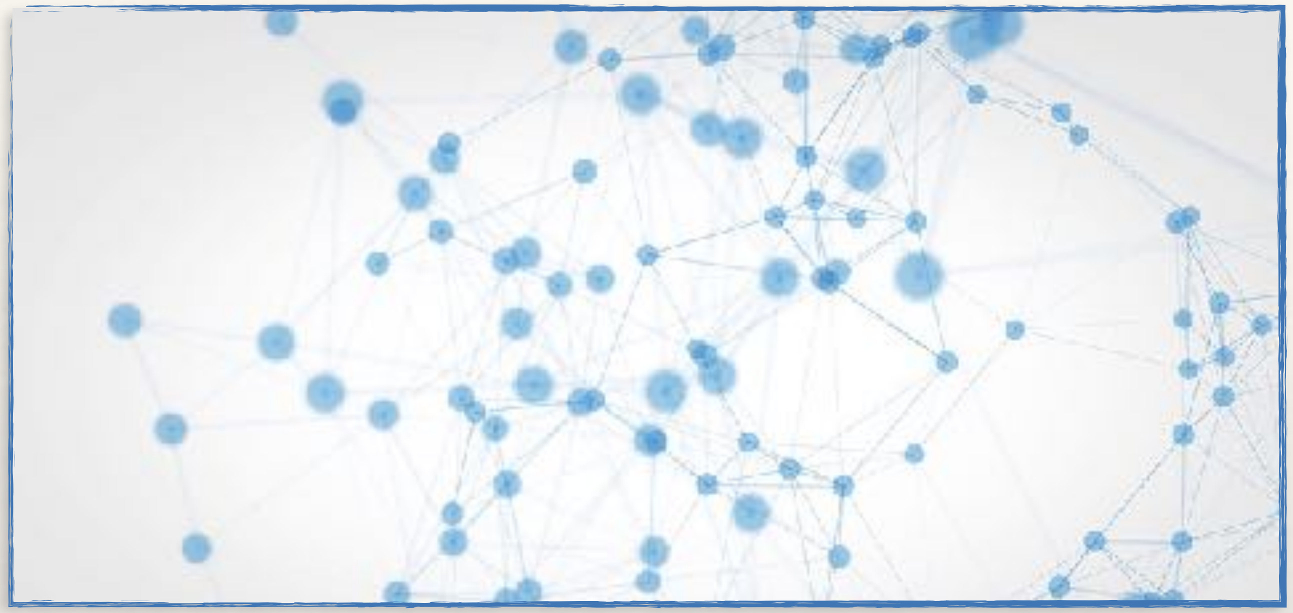
From:

Shape Analysis

To:

Neuroscience
Chemistry
Geography
Shape Analysis
Biophysics
Biology
Network Analysis
Oncology
…

Anything has Shape



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…

Complex Networks

Definition:

A ***network*** is a **complex system** consisting of **individuals** or **entities** connected by specific **ties** such as

- ***Personal Relationship***
- ***Shared Knowledge***
- …

References:

M. Newman, *Networks: An Introduction*, 2010

J. Scott, *Social Network Analysis*, 2017

Complex Networks

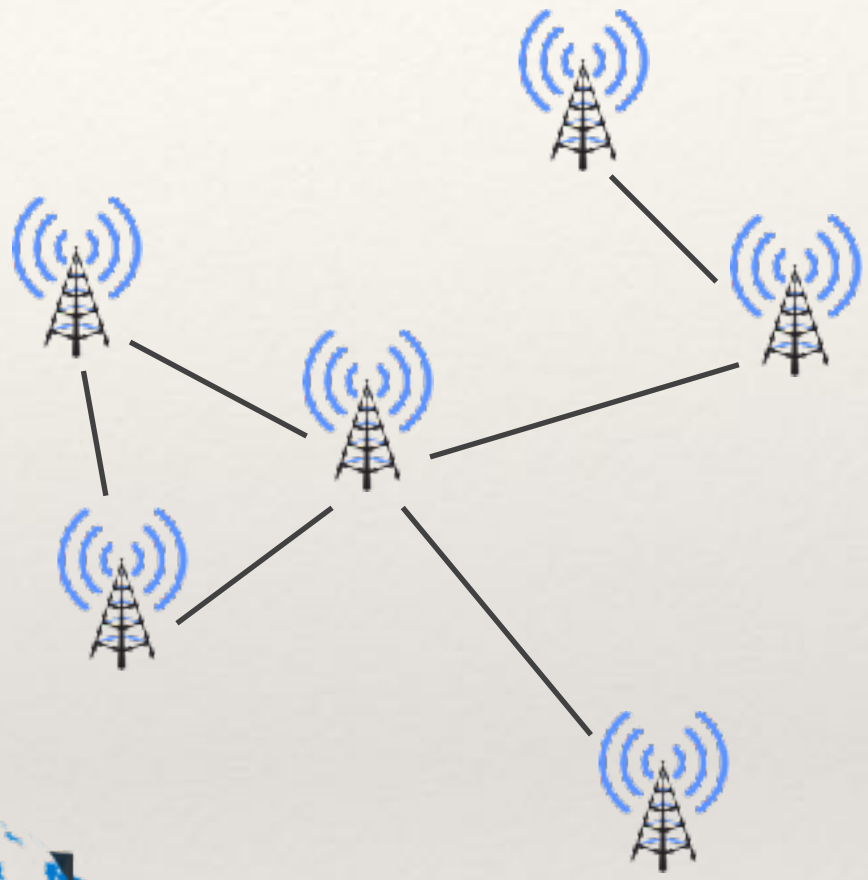
A Bunch of Examples:

- ***Social** Networks*

Complex Networks

A Bunch of Examples:

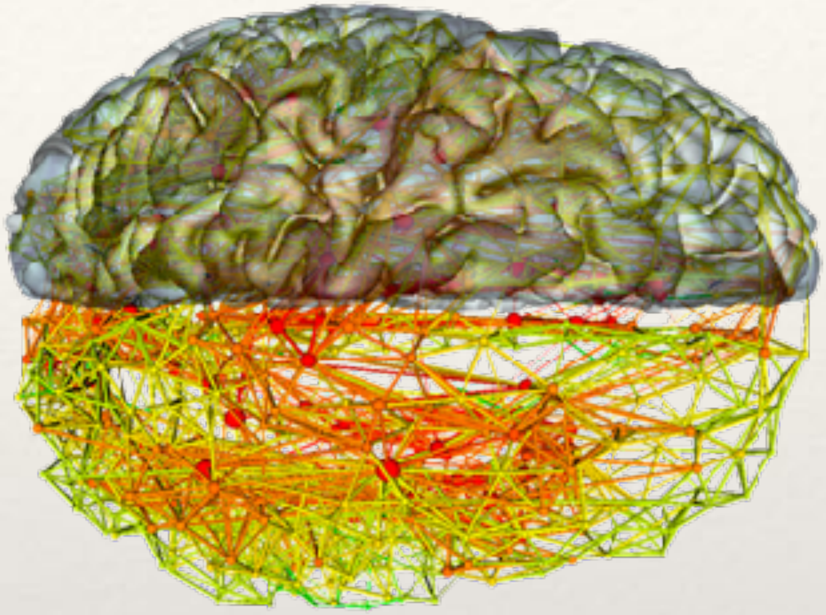
- ***Social*** *Networks*
- ***Sensor*** *Networks*

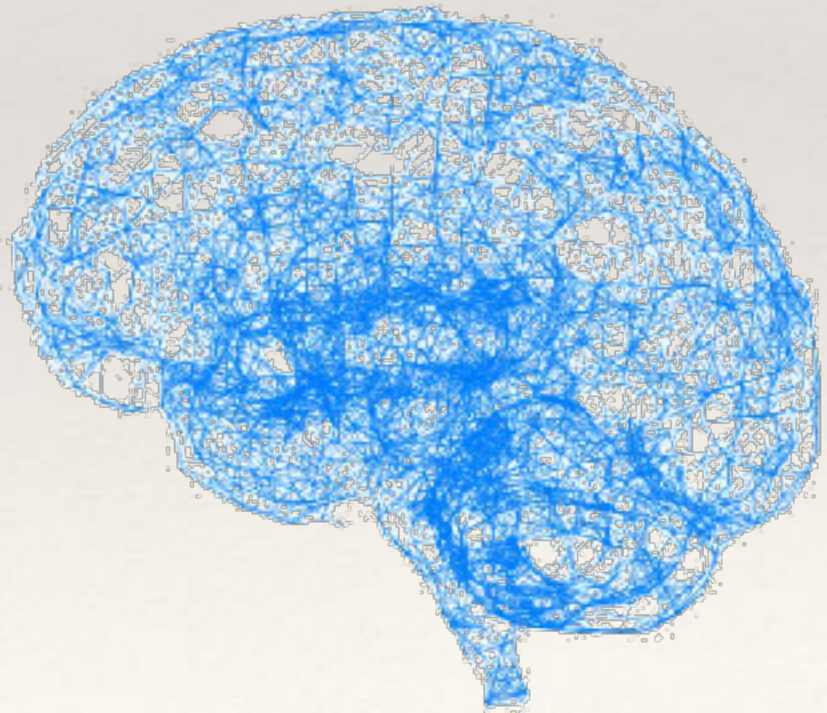


Complex Networks

A Bunch of Examples:

- ***Social*** *Networks*
- ***Sensor*** *Networks*
- ***Biological*** *Networks*





Complex Networks

A Bunch of Examples:

- ***Social*** *Networks*
- ***Sensor*** *Networks*
- ***Biological*** *Networks*
- ***Collaborative*** *Networks*

Complex Networks

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- ***Social*** *Networks*
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- …

Outline

Brief Introduction to
Complex Network
Analysis

Outline

Brief Introduction to Complex Network Analysis

Persistence-based Network Analysis

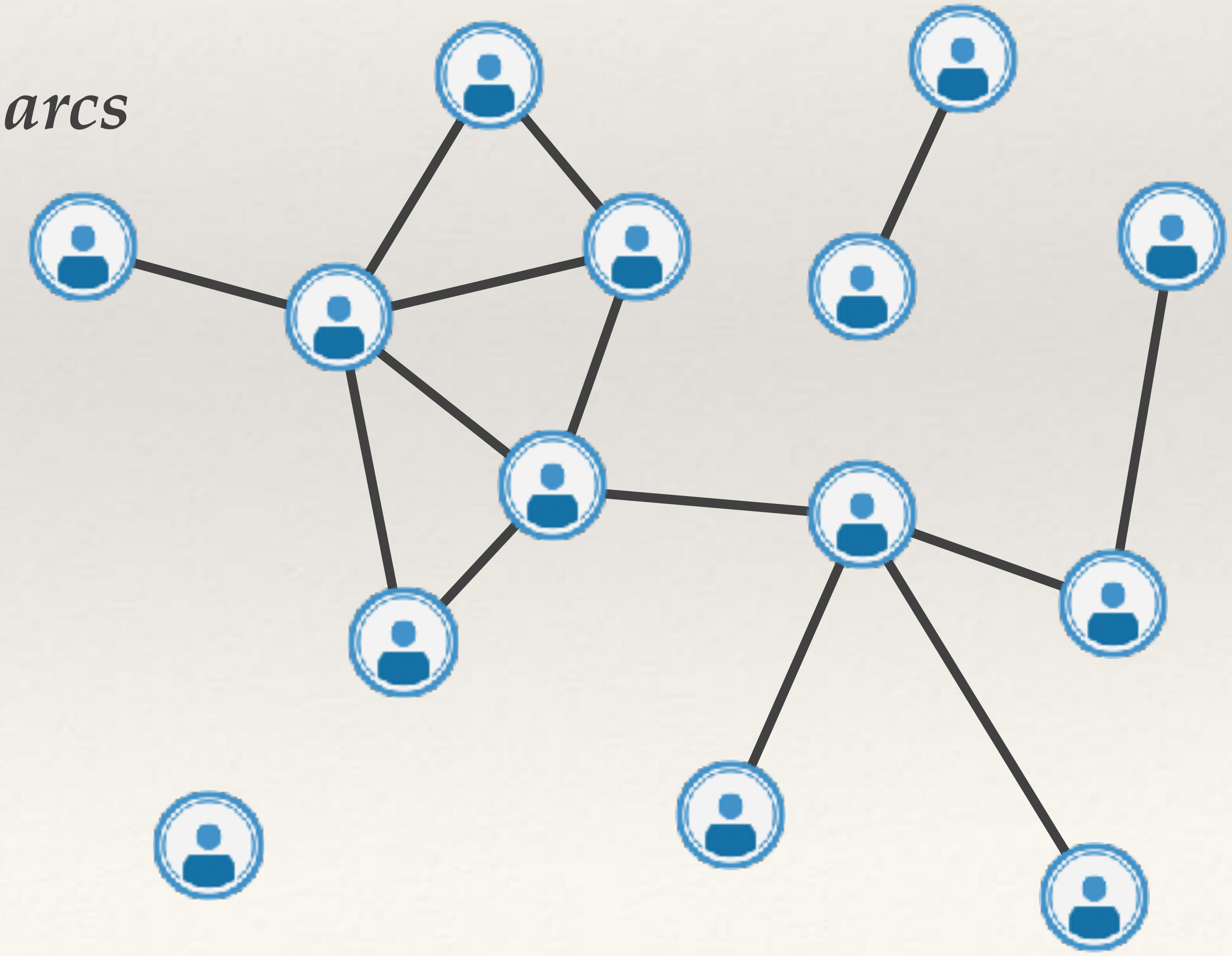
Network Analysis

Representation:

A network can be represented by a **graph** ***G=(V, E)*** such that:

individuals ⟺ *nodes*

ties ⟺ *arcs*



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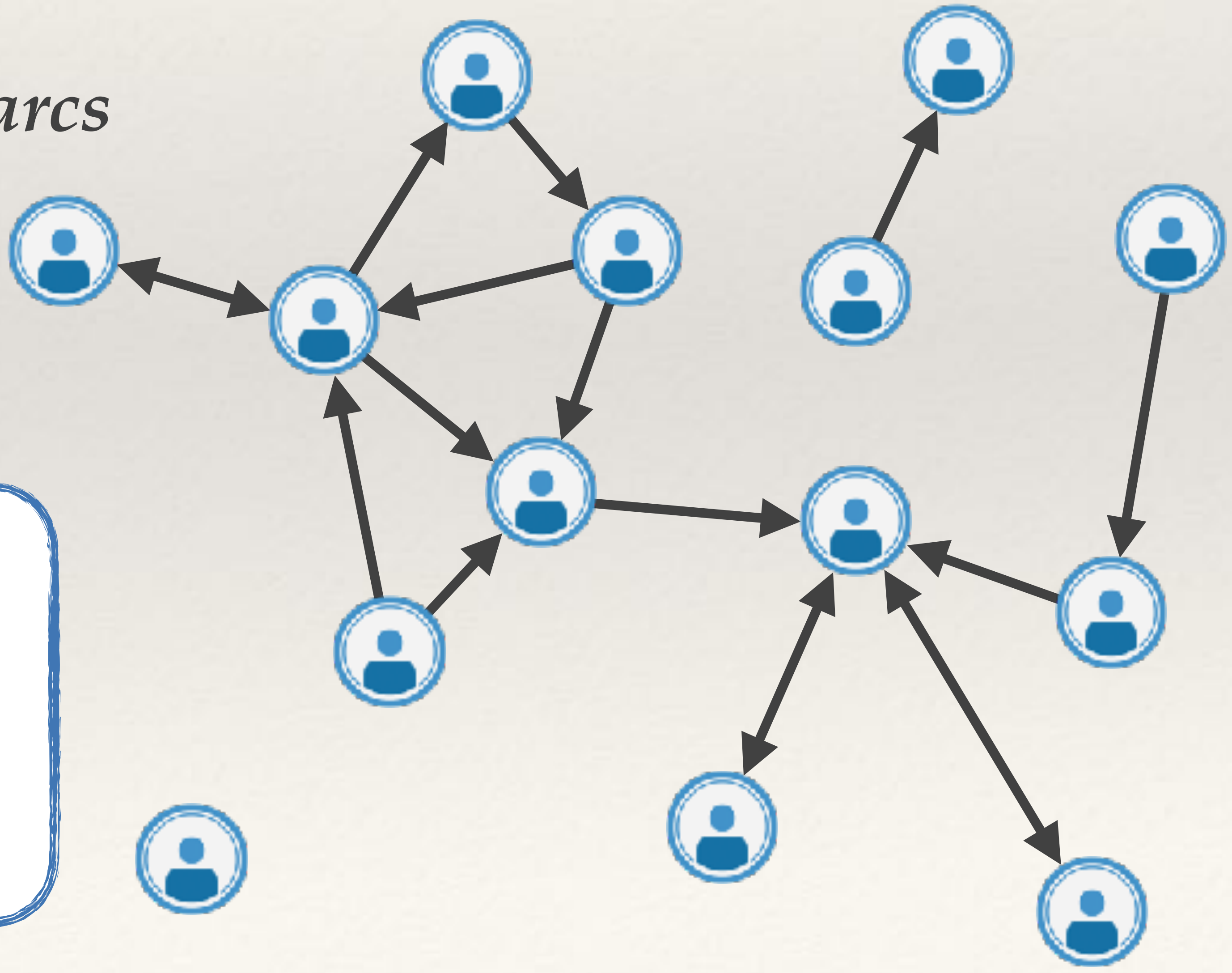

Arcs can be:

- ***directed***
- ***weighted***

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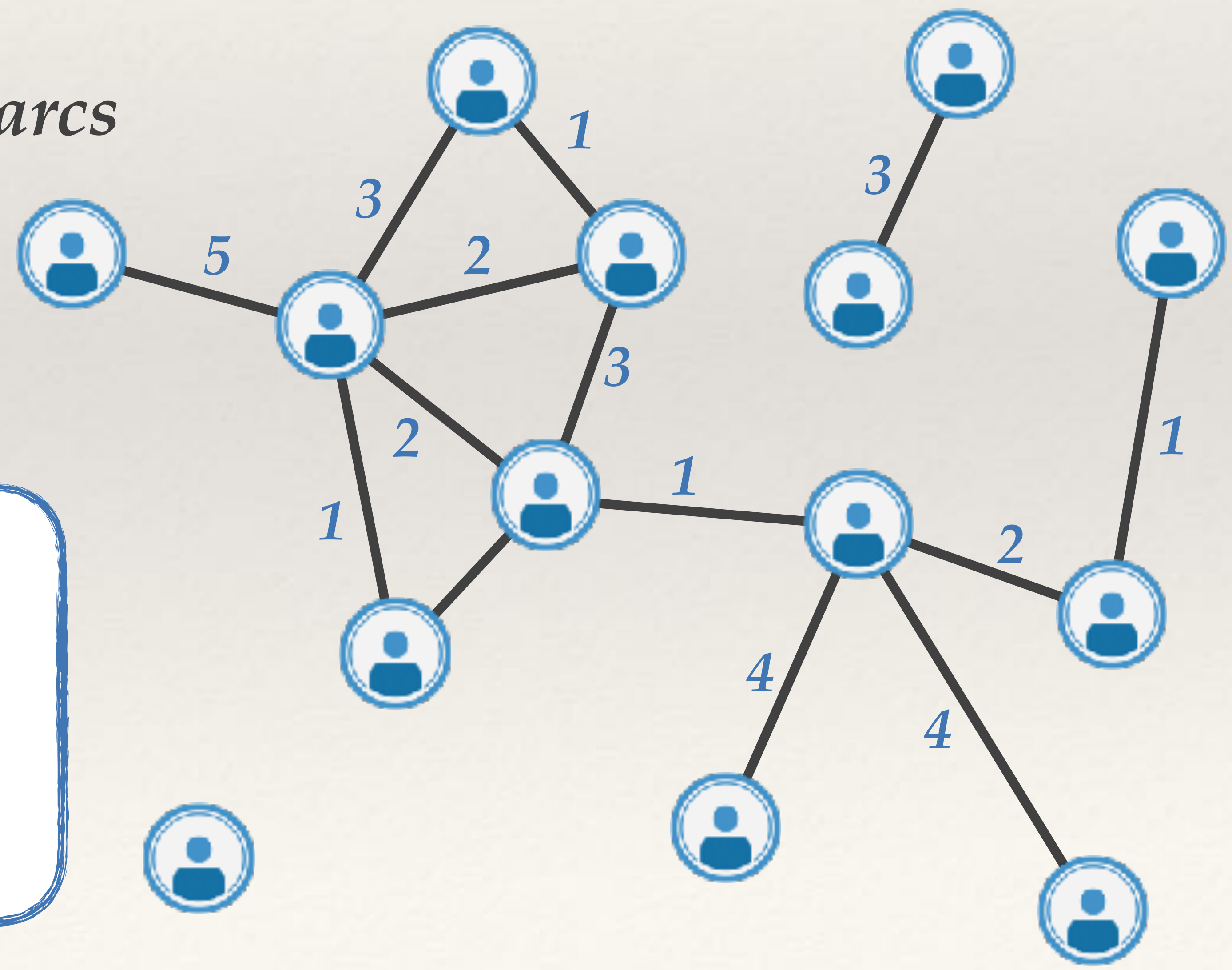

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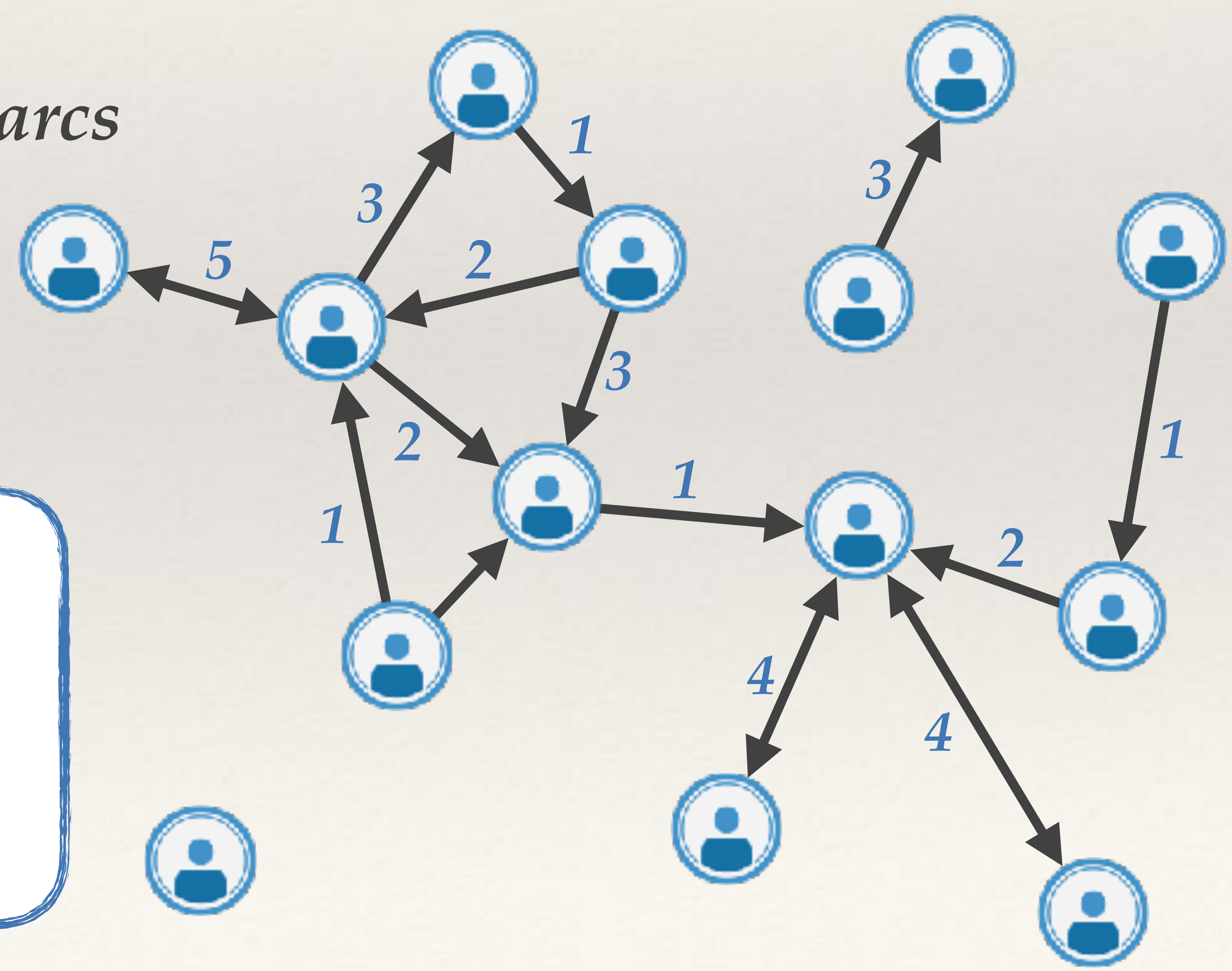

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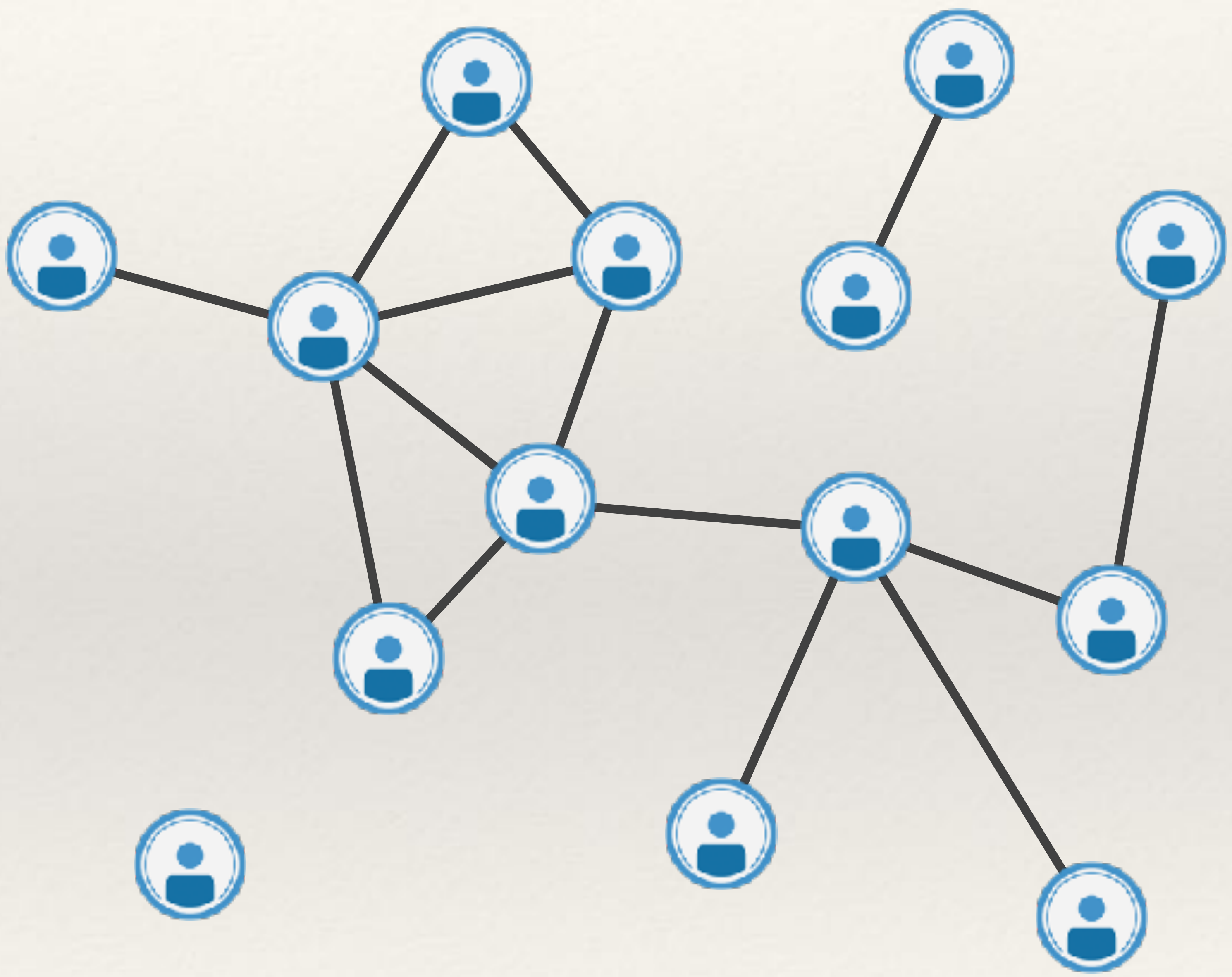
Network Analysis

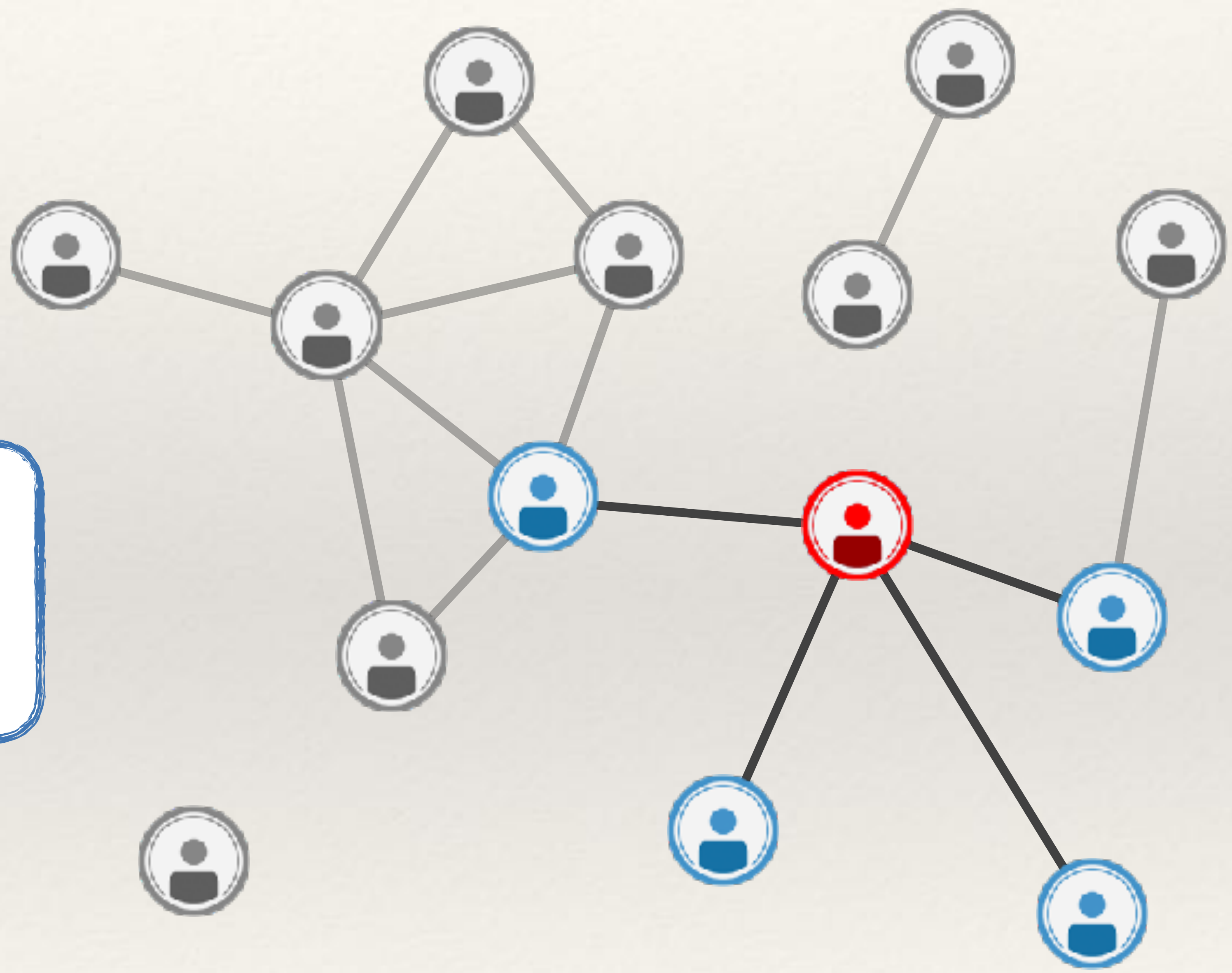
A Two-level Analysis:



Network Analysis

A Two-level Analysis:

- ***Egocentric***
- *Sociocentric*



Network Analysis

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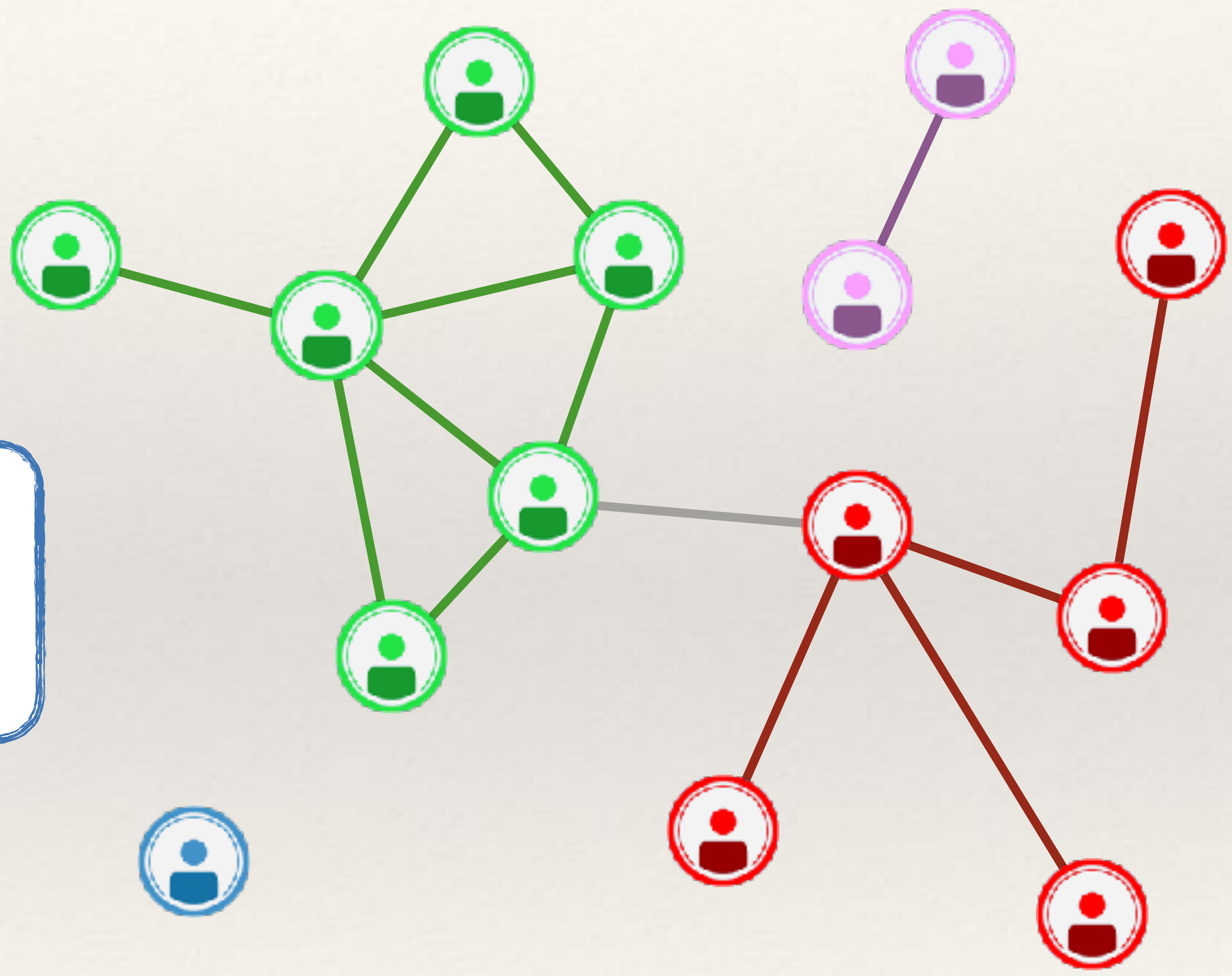
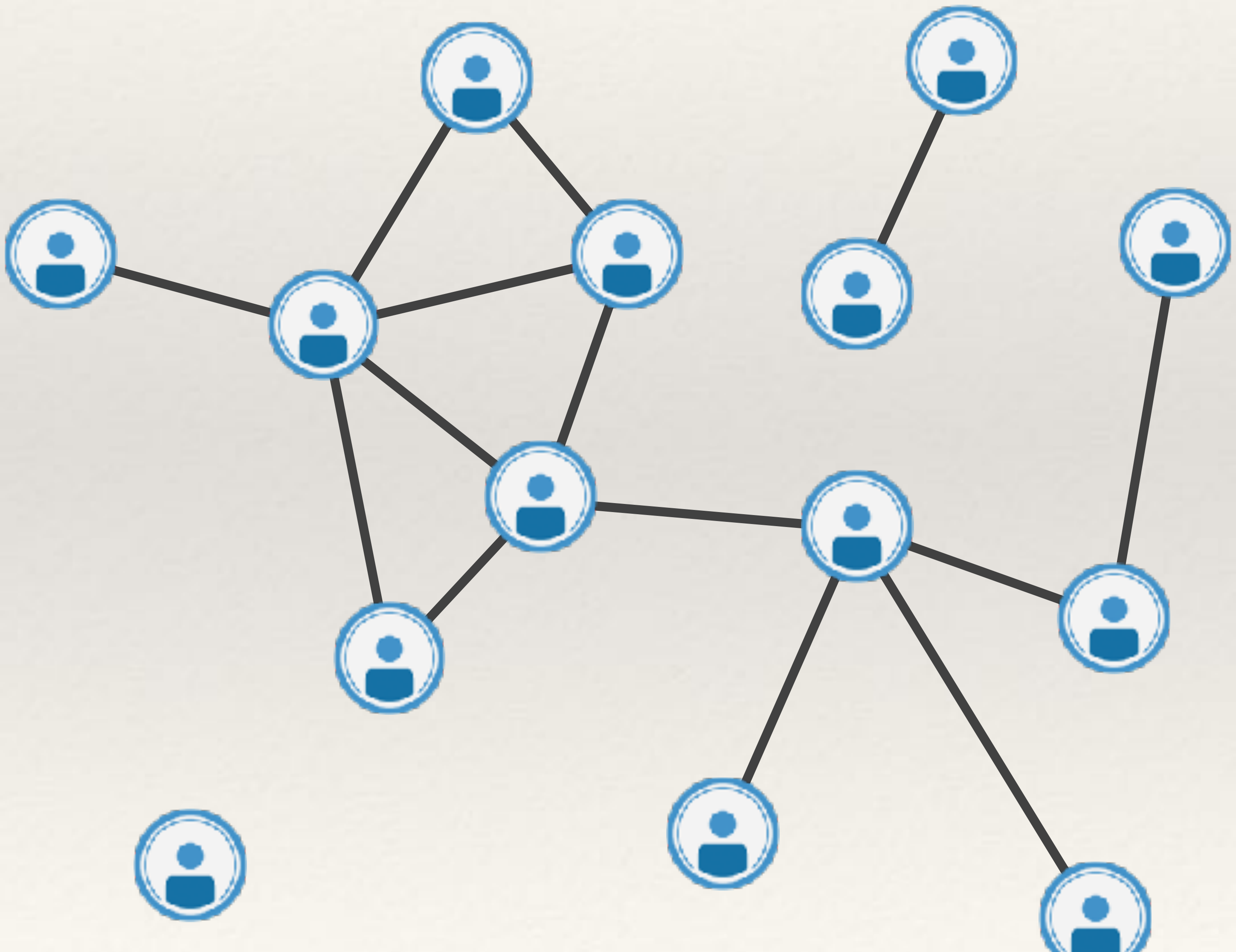


Identifying Key Players

What is the most important individual?



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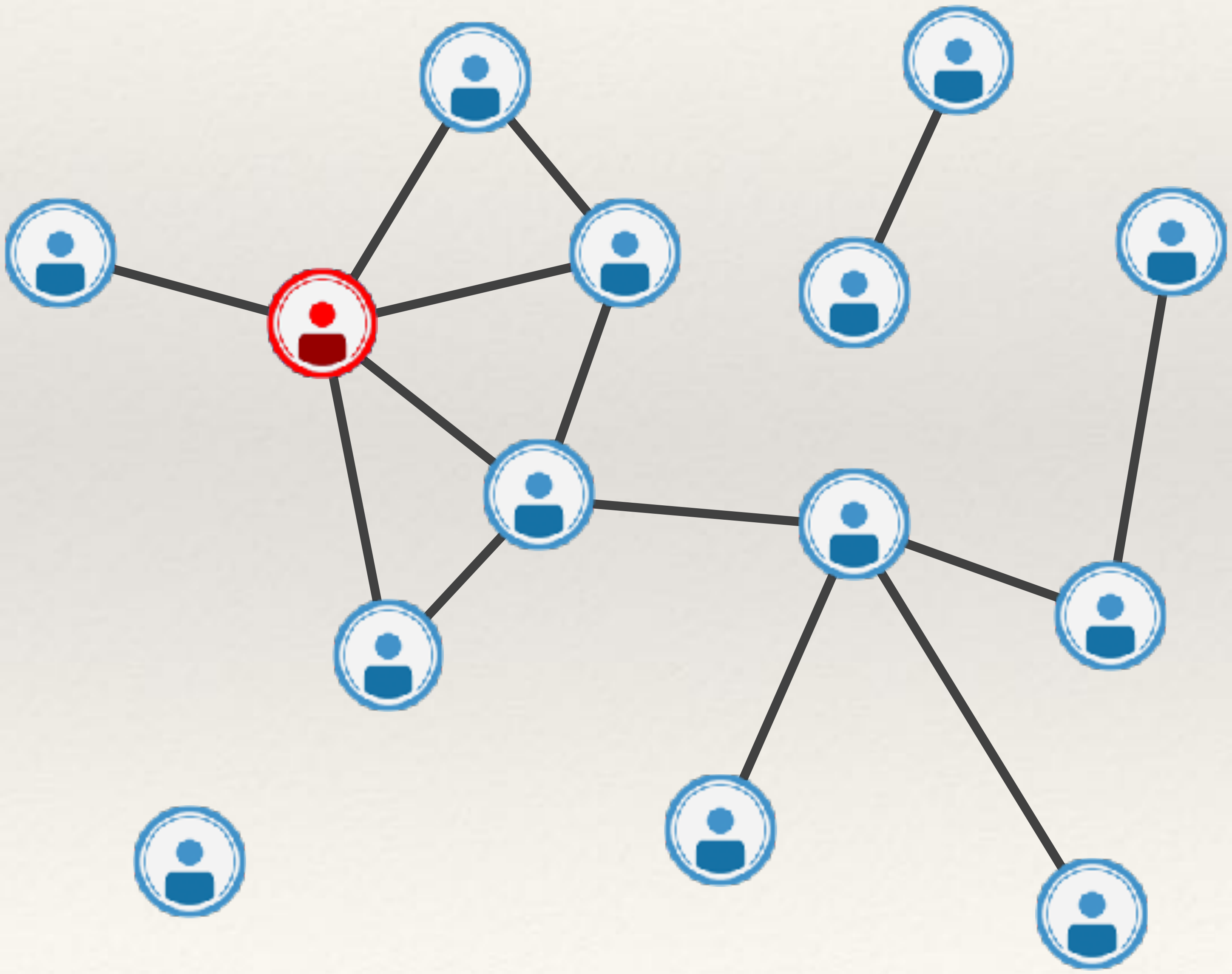

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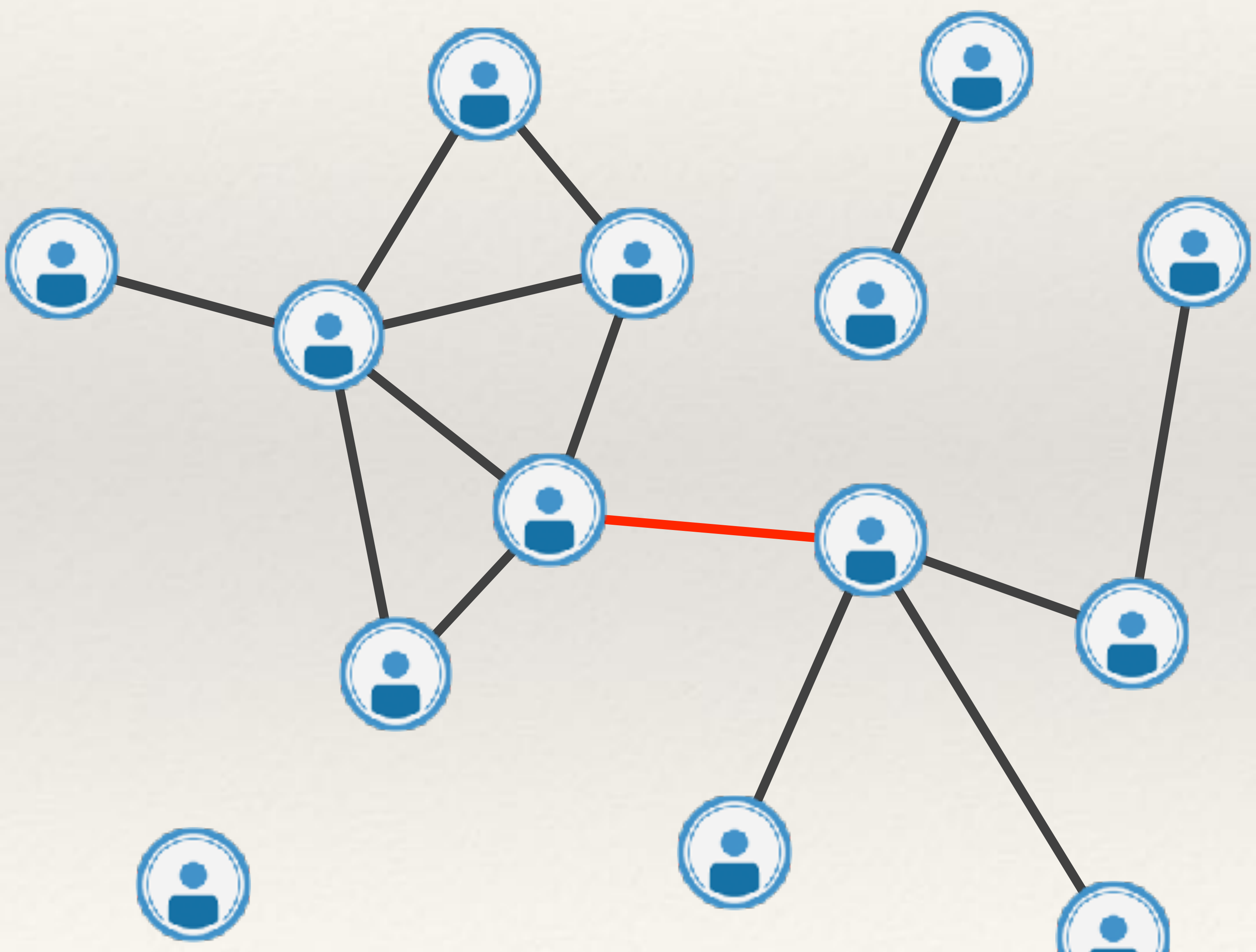

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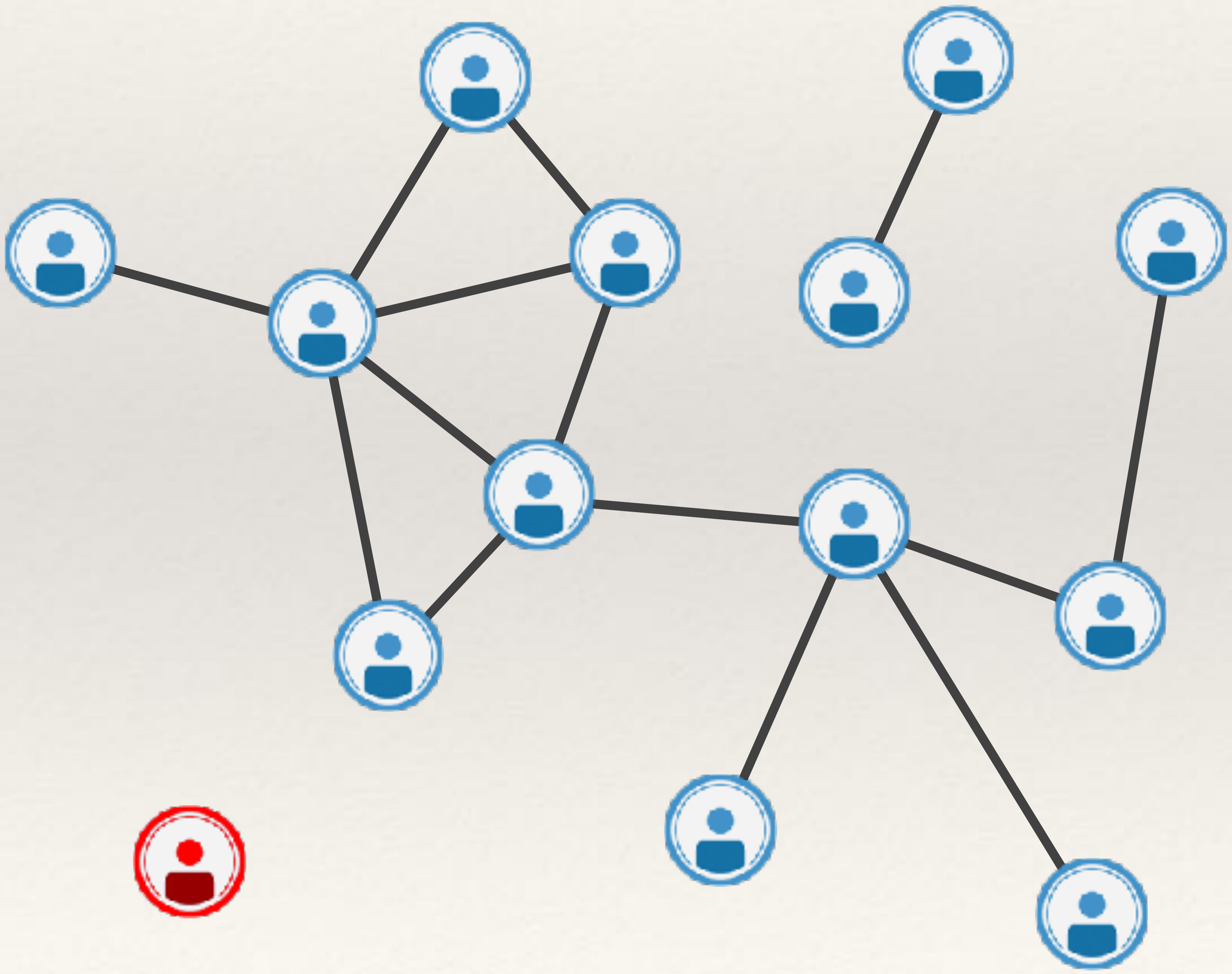


Identifying Key Players

Centrality Measures:

Different criteria to underline **different roles**: $\left\{ \begin{array}{l} \textit{Key players} \\ \textit{Brokers} \\ \textit{Bridges} \\ \textit{Isolated} \\ \ldots \end{array} \right.$

A function $F : V \longrightarrow R$ assigning to each node a ***"centrality"*** value:

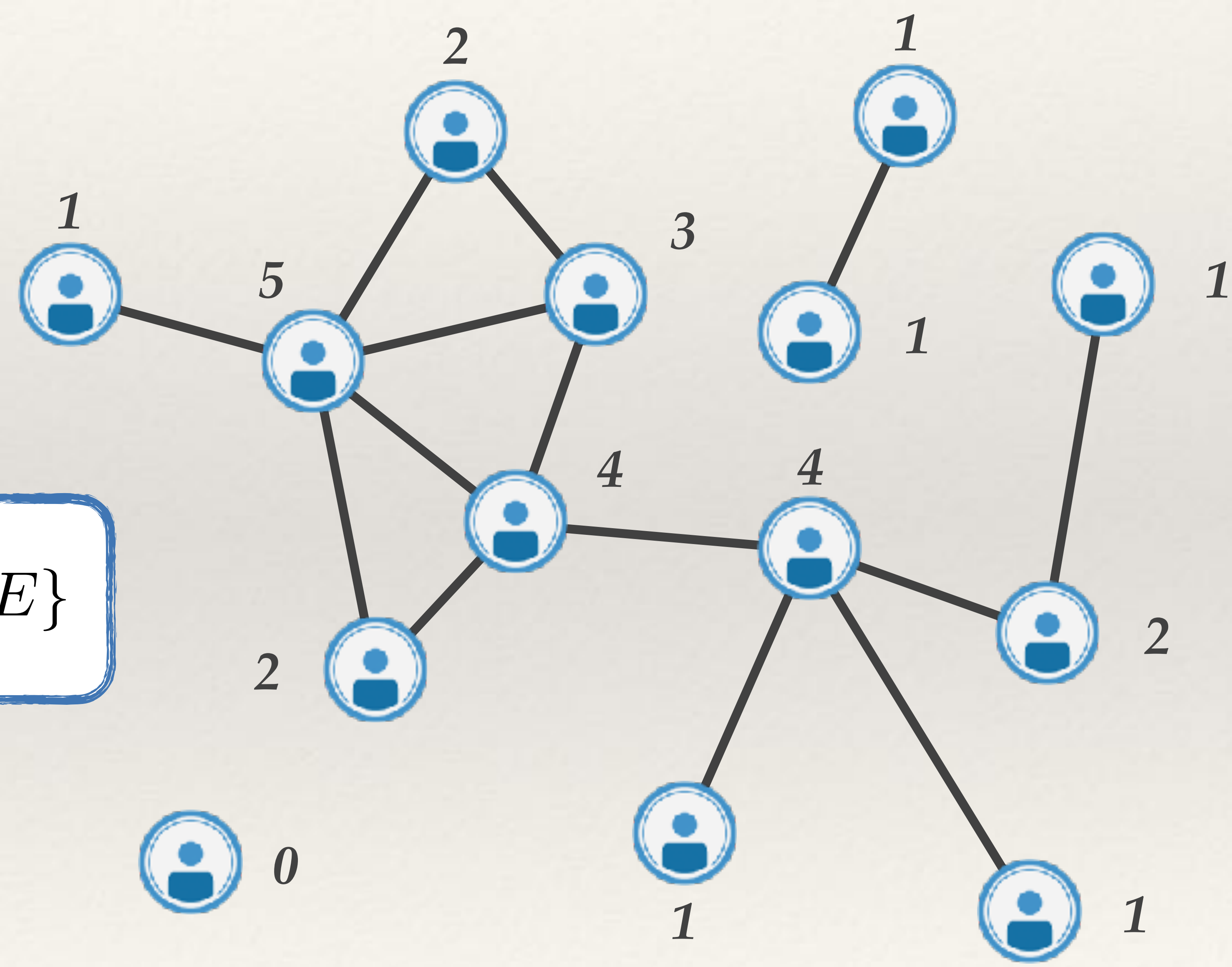
- ***Degree*** *centrality*
- ***Betweenness*** *centrality*
- ***Closeness*** *centrality*
- ***Eigenvector*** *centrality*
- ***Erdös*** *distance*

Identifying Key Players

Degree Centrality:

Given a node v of $G=(V, E)$,

$$D(v) := \#\{u \in V \mid (u, v) \in E\}$$

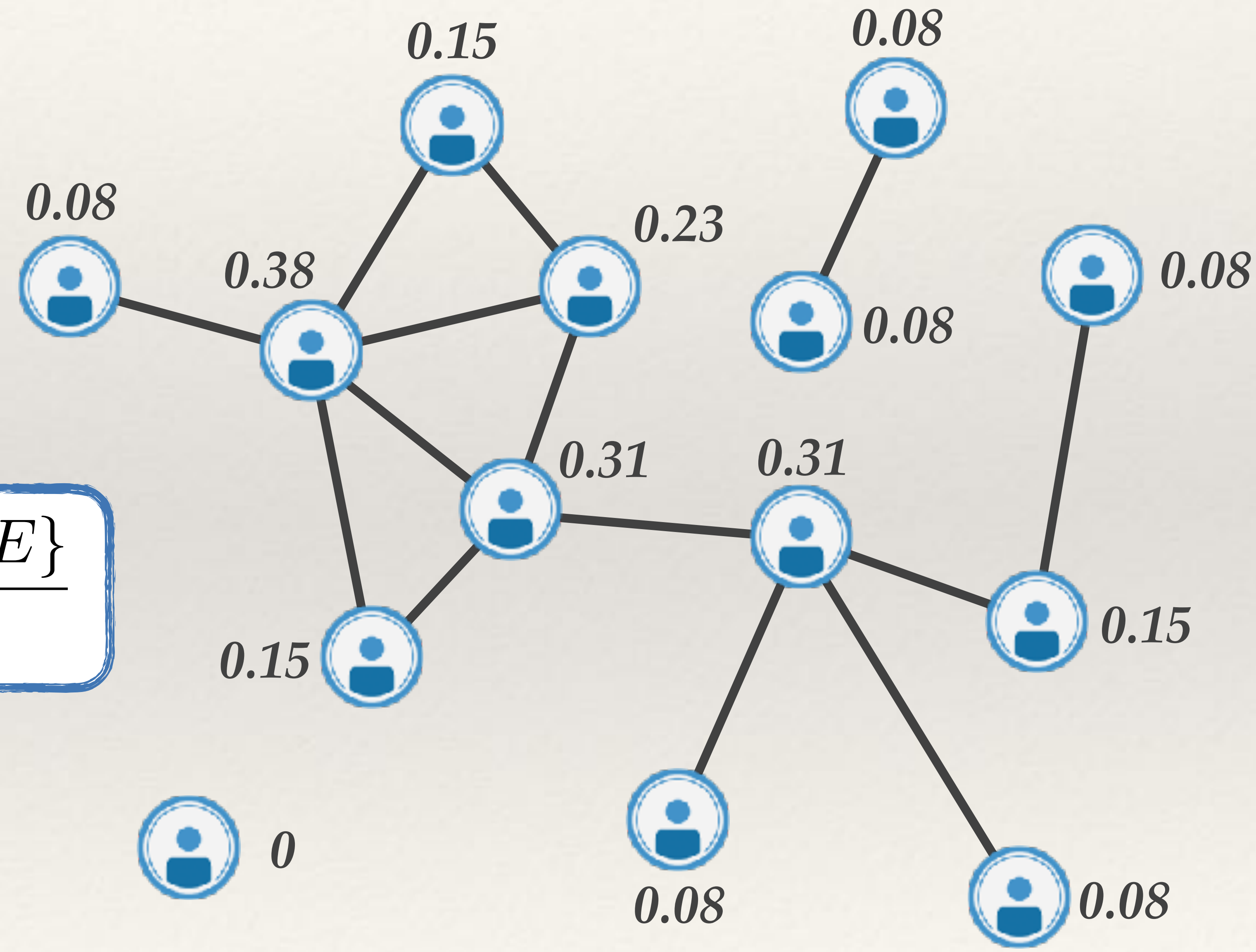


Identifying Key Players

Degree Centrality:

Given a node v of G=(V, E),

$$D(v) := \frac{\#\{u \in V \mid (u,v) \in E\}}{\#V - 1}$$



Identifying Key Players

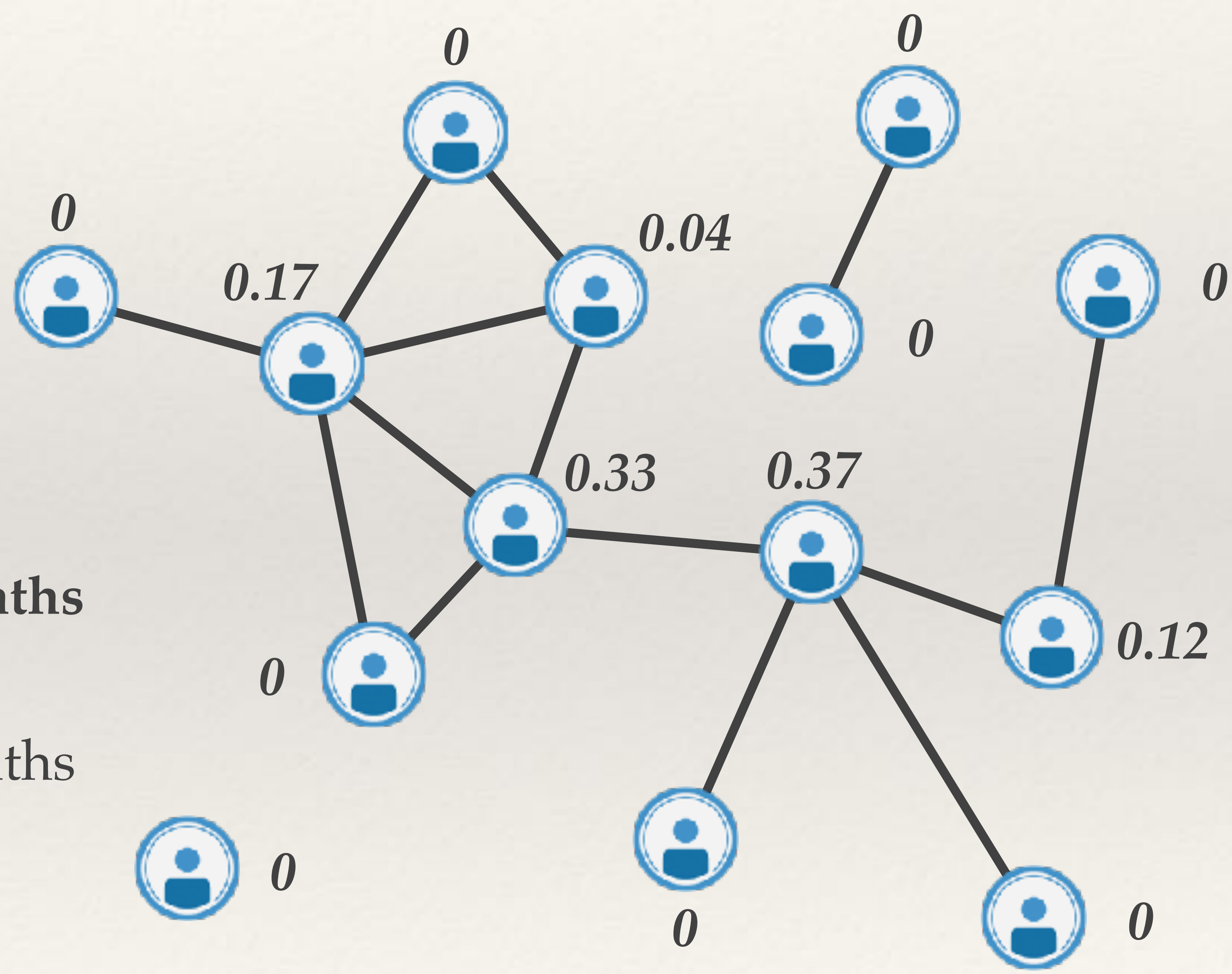
Betweenness Centrality:

Given a node v of G=(V, E),

$$B(v) := \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where:

- σ_{st} is the number of **shortest paths from s to t**
- $\sigma_{st}(v)$ is the number of those paths **passing through v**

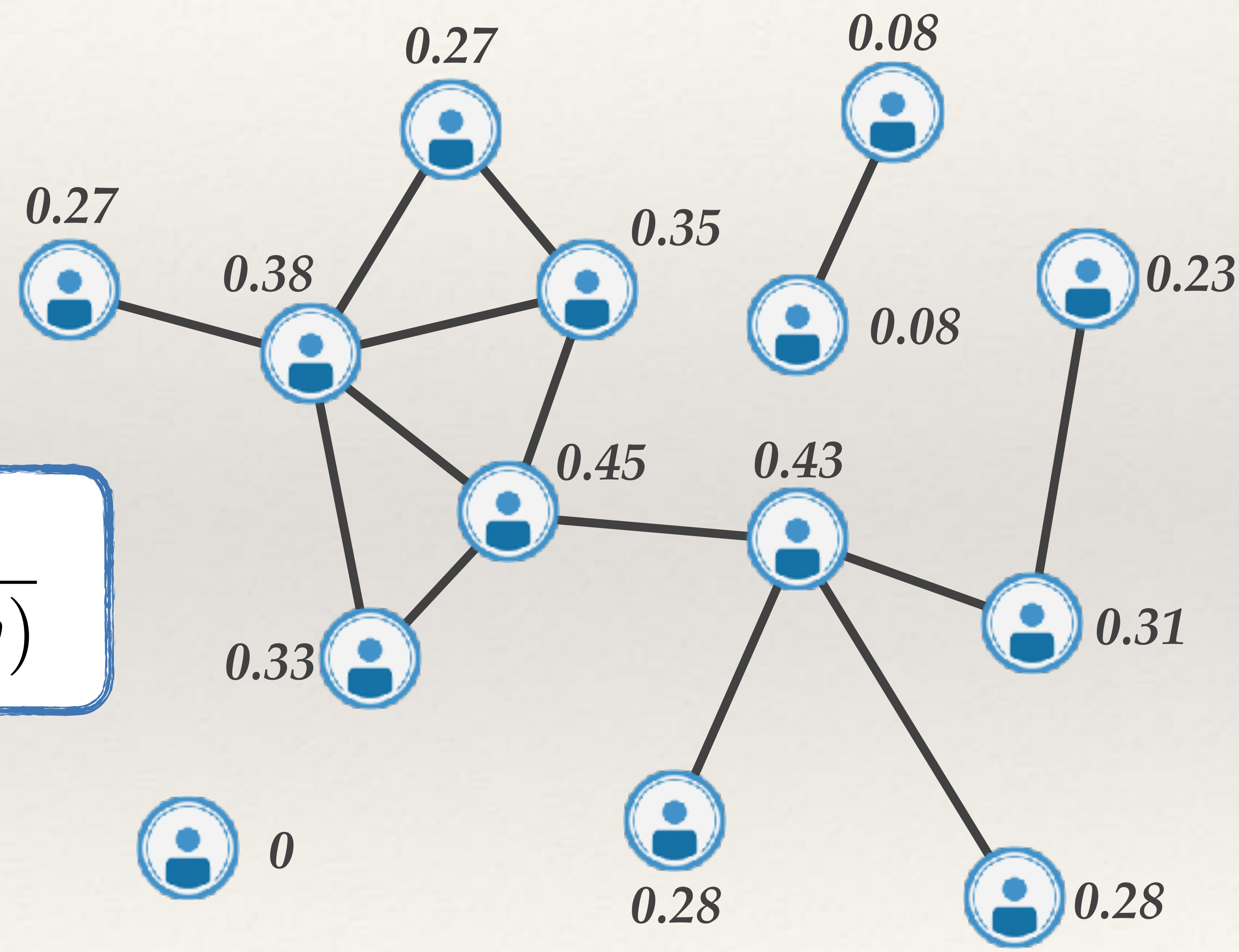


Identifying Key Players

Closeness Centrality:

Given a node v of $G=(V, E)$,

$$C(v) := \frac{\#V - 1}{\sum_{u \in V} d(u, v)}$$



Identifying Key Players

Eigenvector Centrality:

Given a node v of $G=(V, E)$,

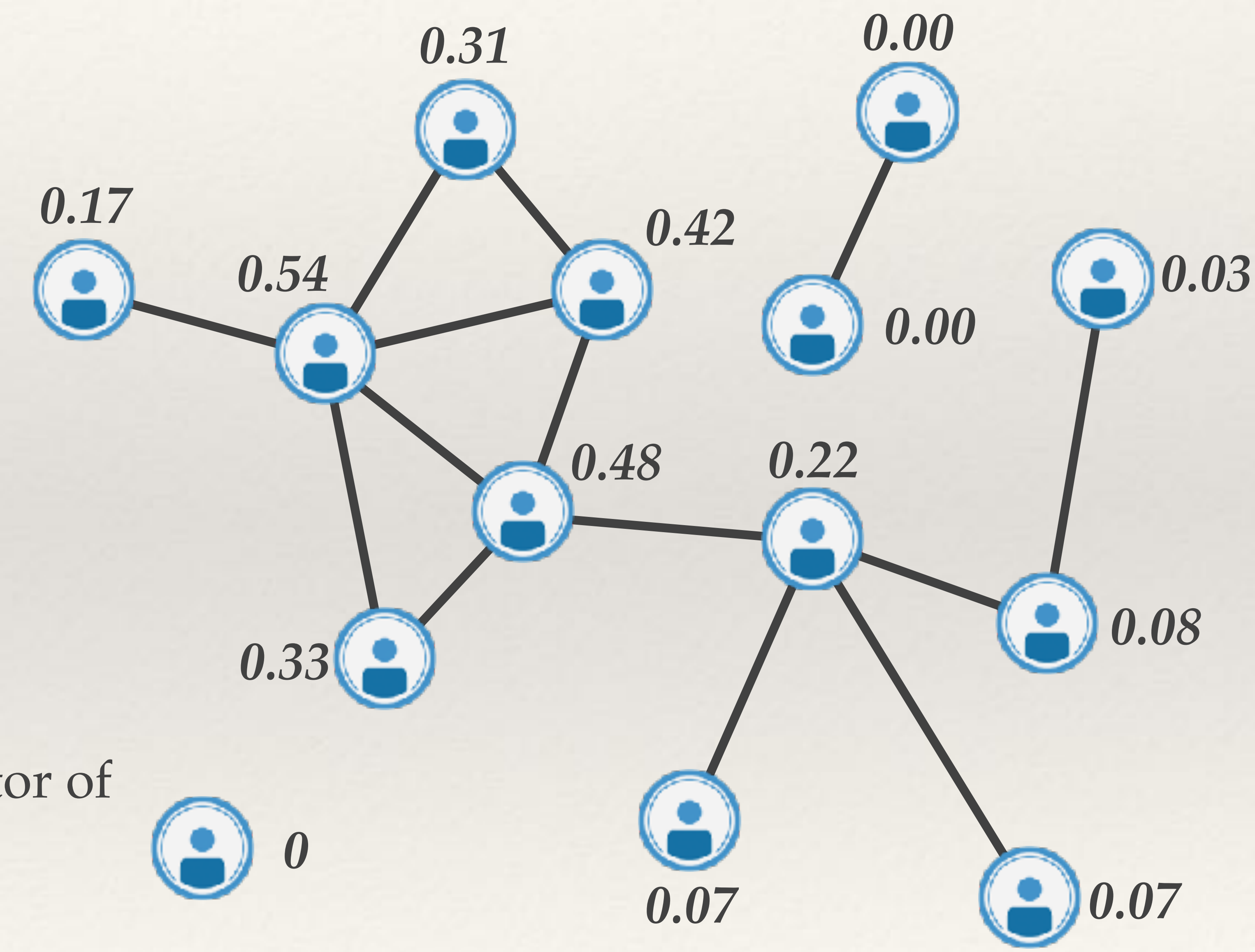
$$x_v := \frac{1}{\lambda} \sum_{u \in V} A_{uv} \, x_u$$

where λ is constant and

$$A_{uv} := \begin{cases} 1 & if(u,v) \in E \\ 0 & otherwise \end{cases}$$

i.e., the v^{th} entry of the eigenvector of

$$\boldsymbol{Ax = \lambda x}$$



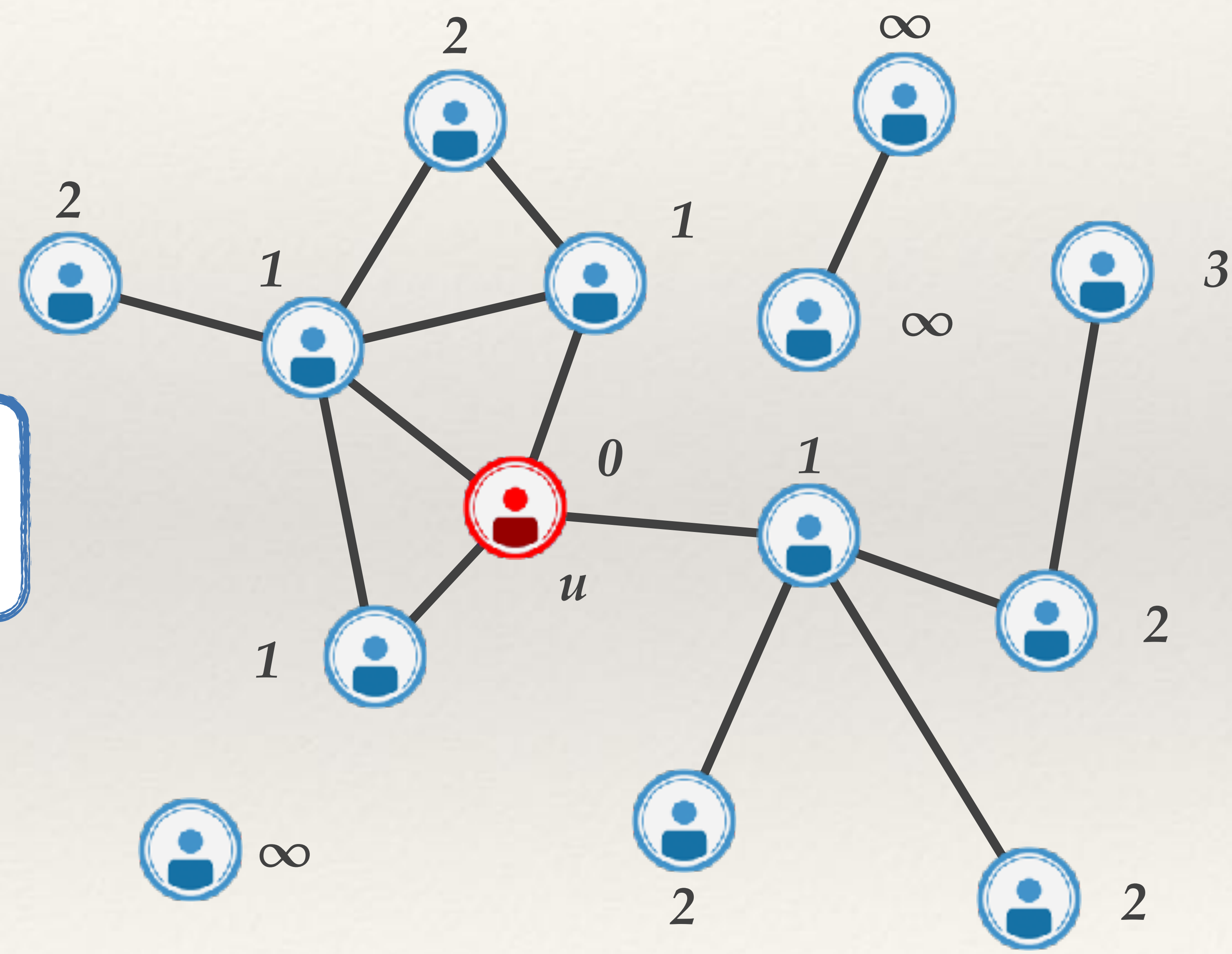
***x*>0** implies λ must be the largest eigenvalue of A and $\boldsymbol{x}$ the corresponding eigenvector

Identifying Key Players

Erdös Distance:

Given two nodes *u*, *v* of *G=(V, E)*,

$$E_u(v) := d(u, v)$$



Named after ***Paul Erdös***,

- one of the most prolific mathematicians of the 20th century

Identifying Key Players

Centrality Measures:

A centrality measure for ***any query***

Degree	***How many individuals can v reach directly?***
Betweenness	***How likely is v to be the most direct route between two individuals?***
Closeness	***How fast can v reach everyone in the network?***
Eigenvector	***How well is v connected to other well-connected individuals?***
Erdös	***How far is v from a specific individual?***

Structural Analysis

Sociocentric Networks:

- **Structural Metrics:**
  - *Average of a Centrality Measure*
  - *Diameter*
  - *Density*
  - *Transitivity*
  - …
- **Community Decomposition:**
  - *Atomic Communities*
  - *Clustering Techniques*

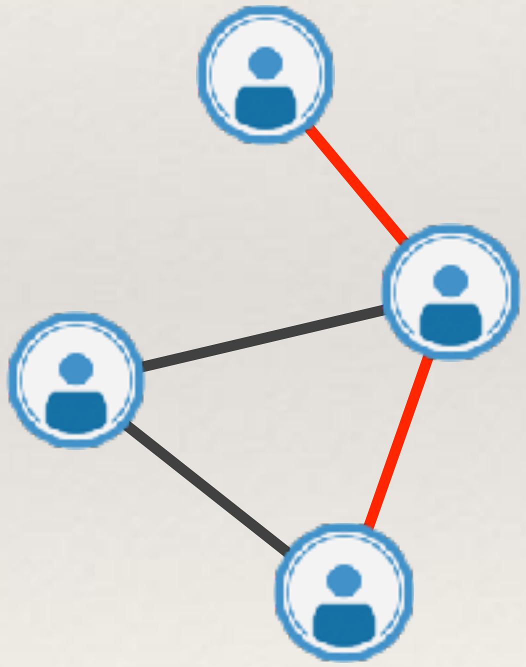
Structural Analysis

Structural Metrics:

- *How far are two individuals at most?*

Diameter:

The longest shortest path between any two nodes



$Diameter(G) = 2$

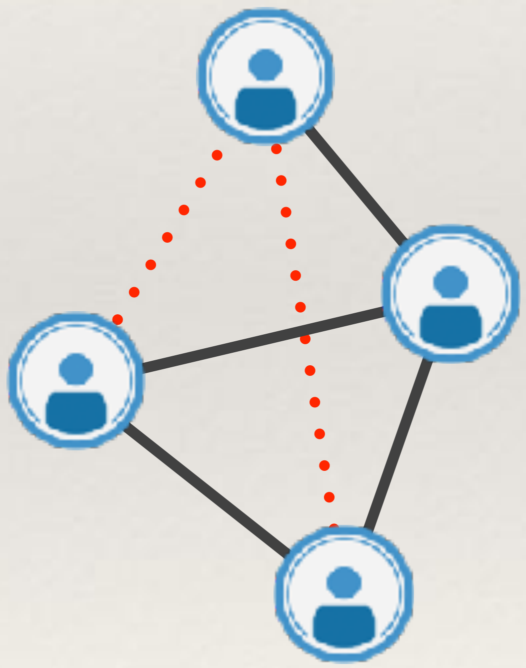
Structural Analysis

Structural Metrics:

- *How close is G to being an "everyone knows everyone" network?*

Density:

$$\frac{\textit{Number of edges of G}}{\textit{Number of all possible edges}}$$



Density(G) = 4/6= 0.67

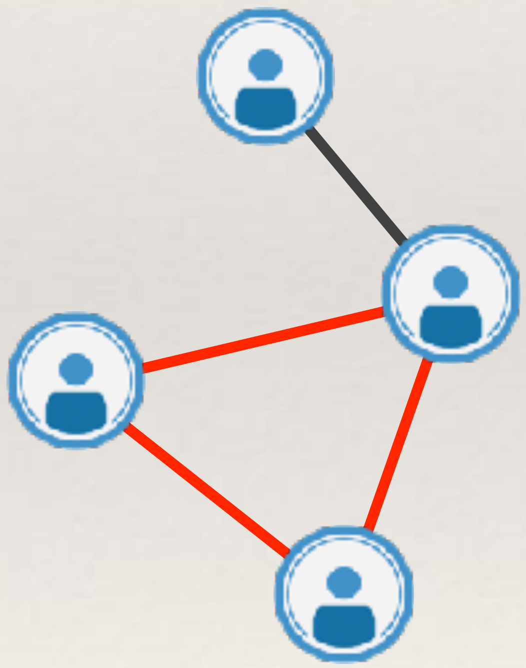
Structural Analysis

Structural Metrics:

- *How likely are two individuals connected to an individual v to be connected to each other?*

Transitivity:

$$\frac{\textit{Number of closed triplets of nodes}}{\textit{Number of connected triplets}}$$



Transitivity(G) = 1/3 = 0.33

Structural Analysis

Community Decomposition:

- *Atomic Communities:*
  - *Clique*
  - *n-Clique*
  - *n-Clan*
  - *n-Club*
  - *k-Plex*
  - *k-Core*
  - …

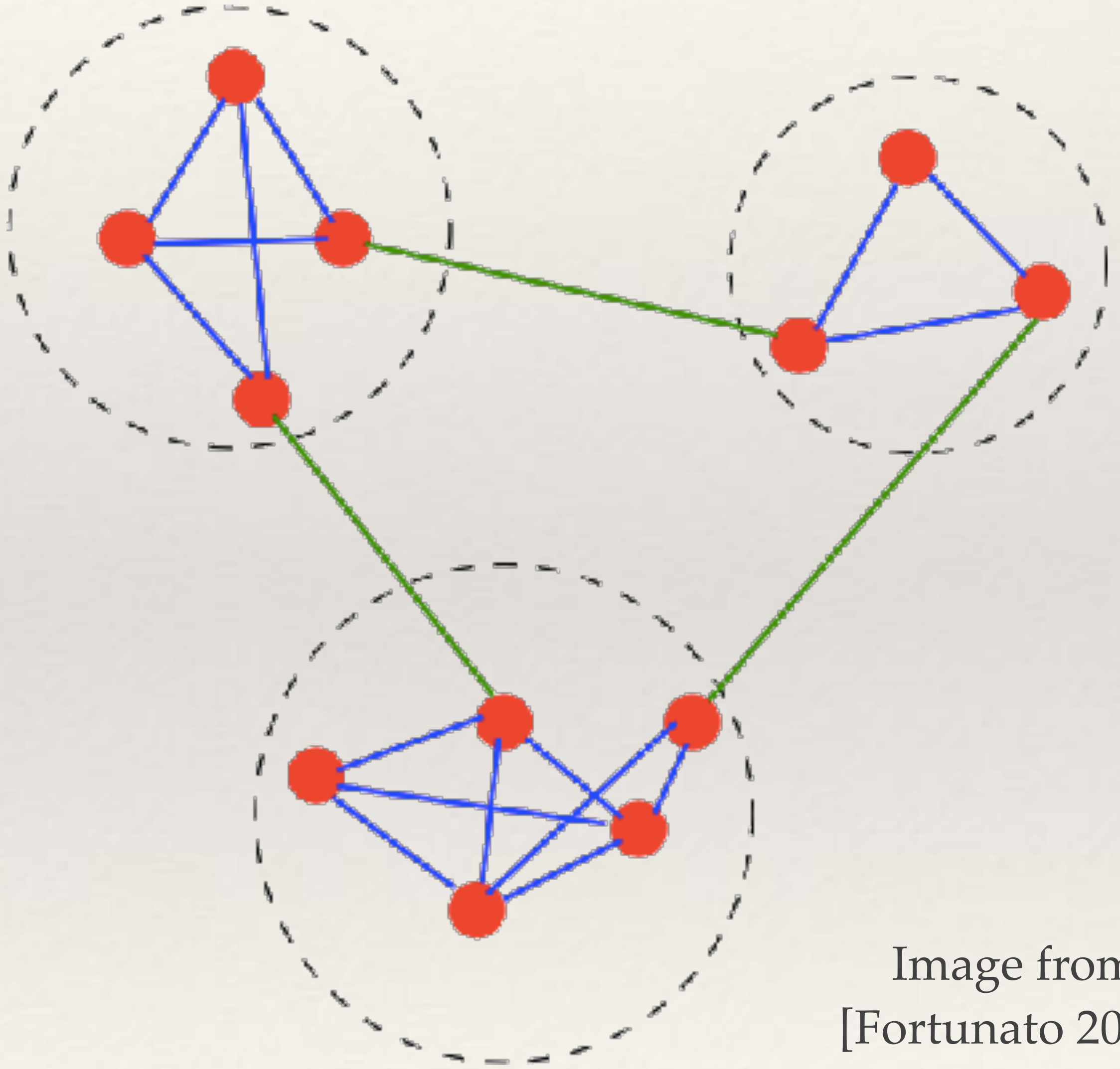


Image from [Fortunato 2009]

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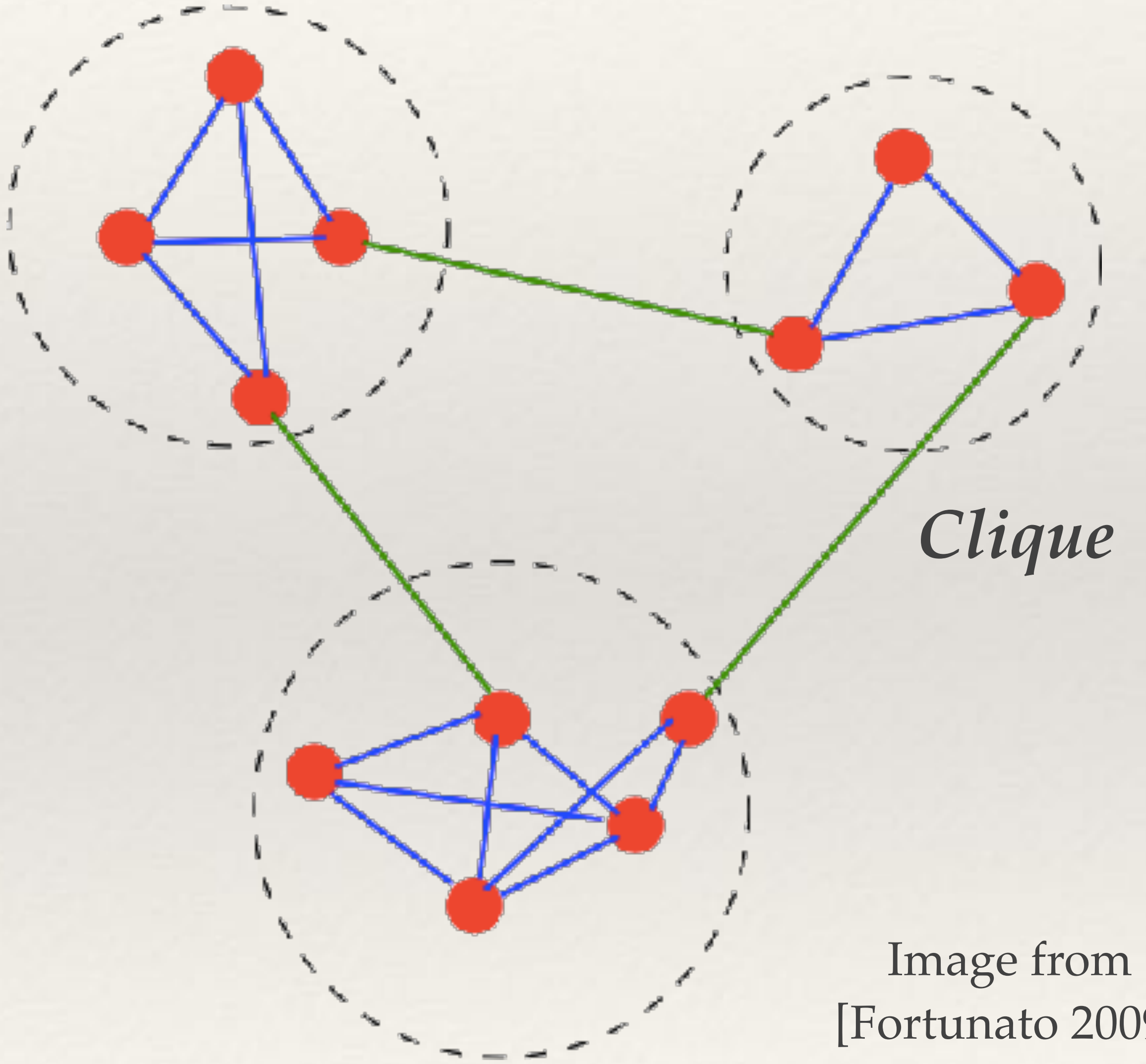

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Clique:

maximal subgraph whose nodes are all adjacent to each other

Structural Analysis

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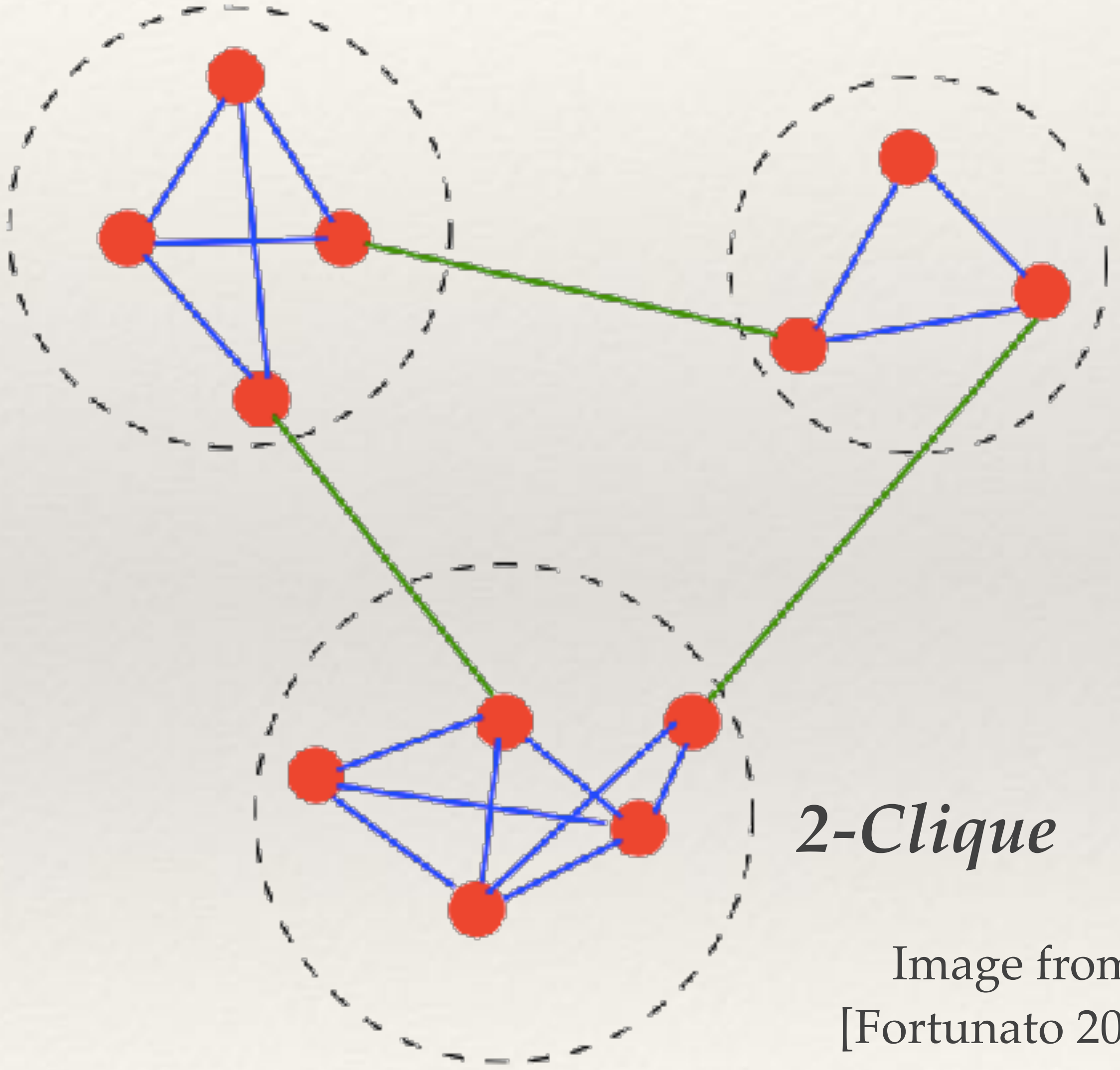

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***n*-Clique:**

maximal subgraph such that the distance of each pair of its nodes is not greater than *n*

Structural Analysis

Community Decomposition:

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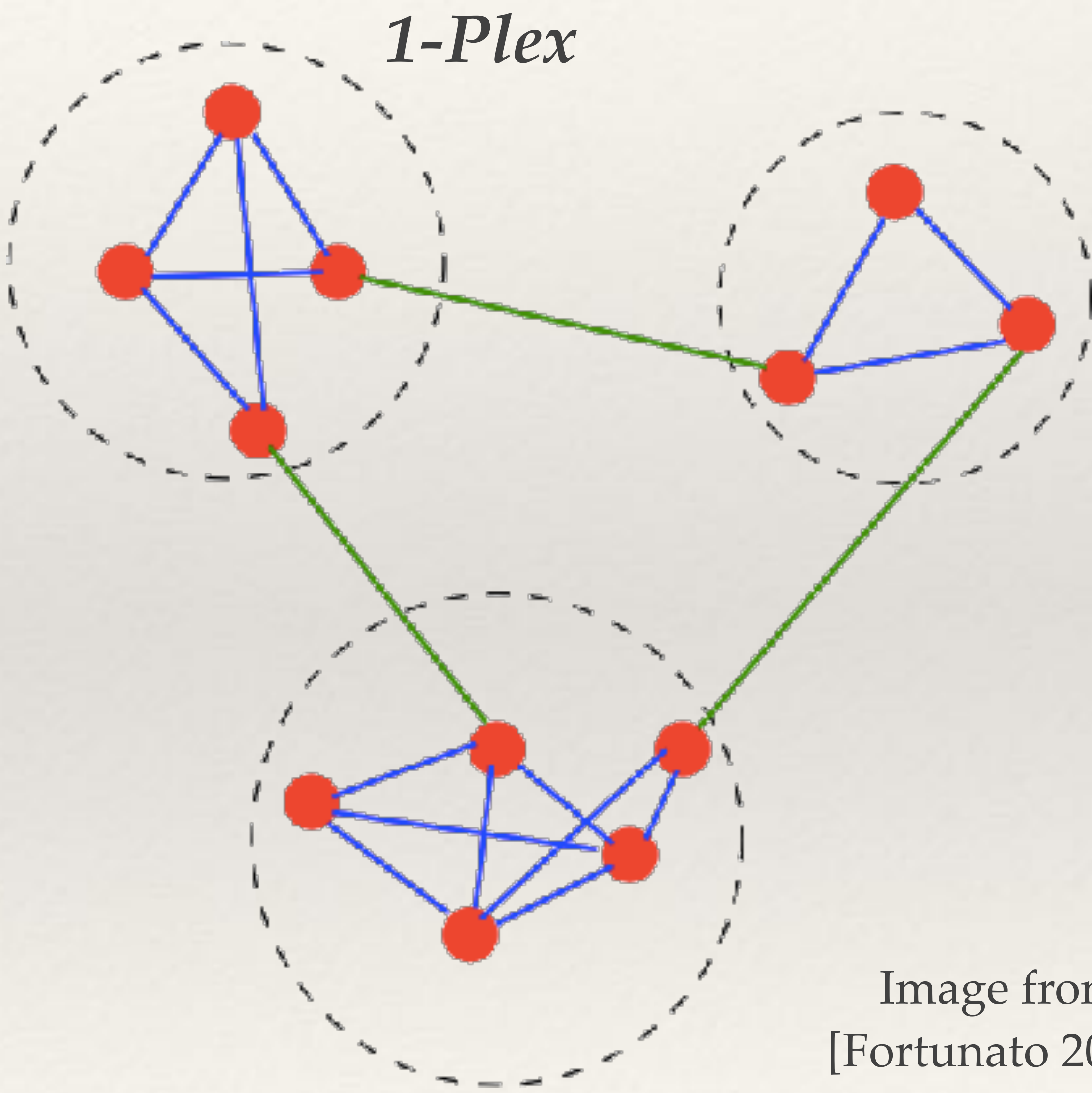

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***k*-Plex:**

maximal subgraph in which each node is adjacent to all other nodes of the subgraph except at most *k* of them

Structural Analysis

Clustering Techniques:

Agglomerative (bottom-up)

Divisive (top-dow)

approach based on

- Centrality Measures
- Atomic Communities
- Quality Functions

Structural Analysis

Clustering Techniques:

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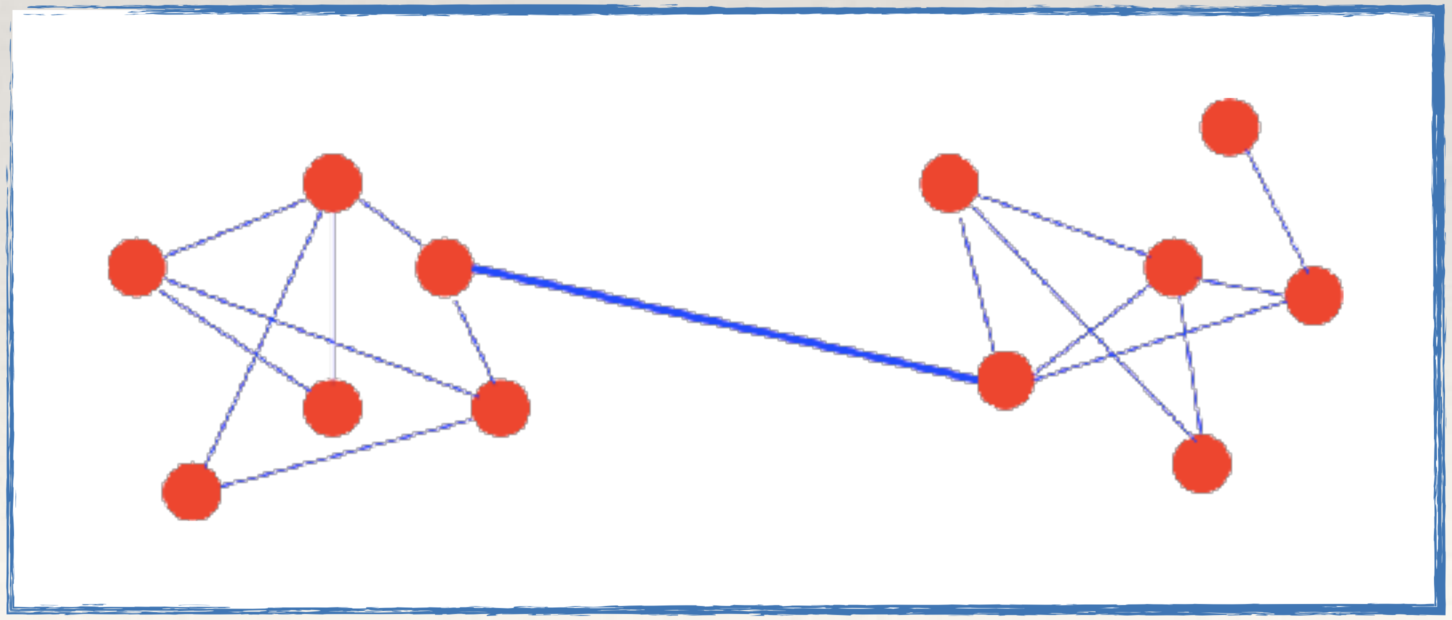
Girvan-Newman Algorithm:

Iterated removal of the edge with largest ***betweenness centrality***



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Structural Analysis

Clustering Techniques:

Agglomerative (bottom-up)	approach based on	Centrality Measures
		Atomic Communities
Divisive (top-dow)		Quality Functions

Clique Percolation:

***k-adjacency*:** two clique of size *k* are *k*-adjacent if they share *k-1* nodes

***k-clique community*:** maximal union of cliques of size *k* pairwise connected by a sequence of *k*-adjacent cliques

Decomposition in *k*-clique communities

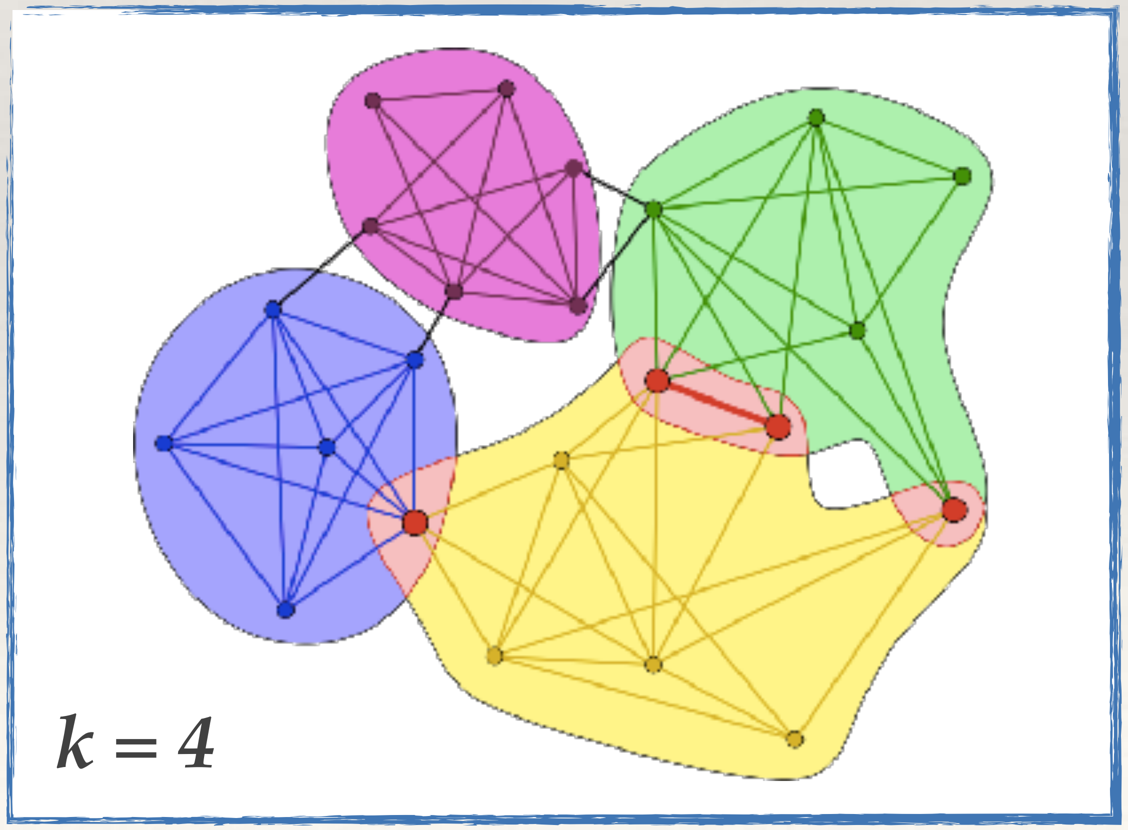

Image from [Palla et al. 2005]

Structural Analysis

Clustering Techniques:

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Atomic Communities

Quality Functions

Modularity-based Algorithm:

Modularity: measure for clustering quality

Iterated aggregation of communities of nodes whose merging ***increases modularity***

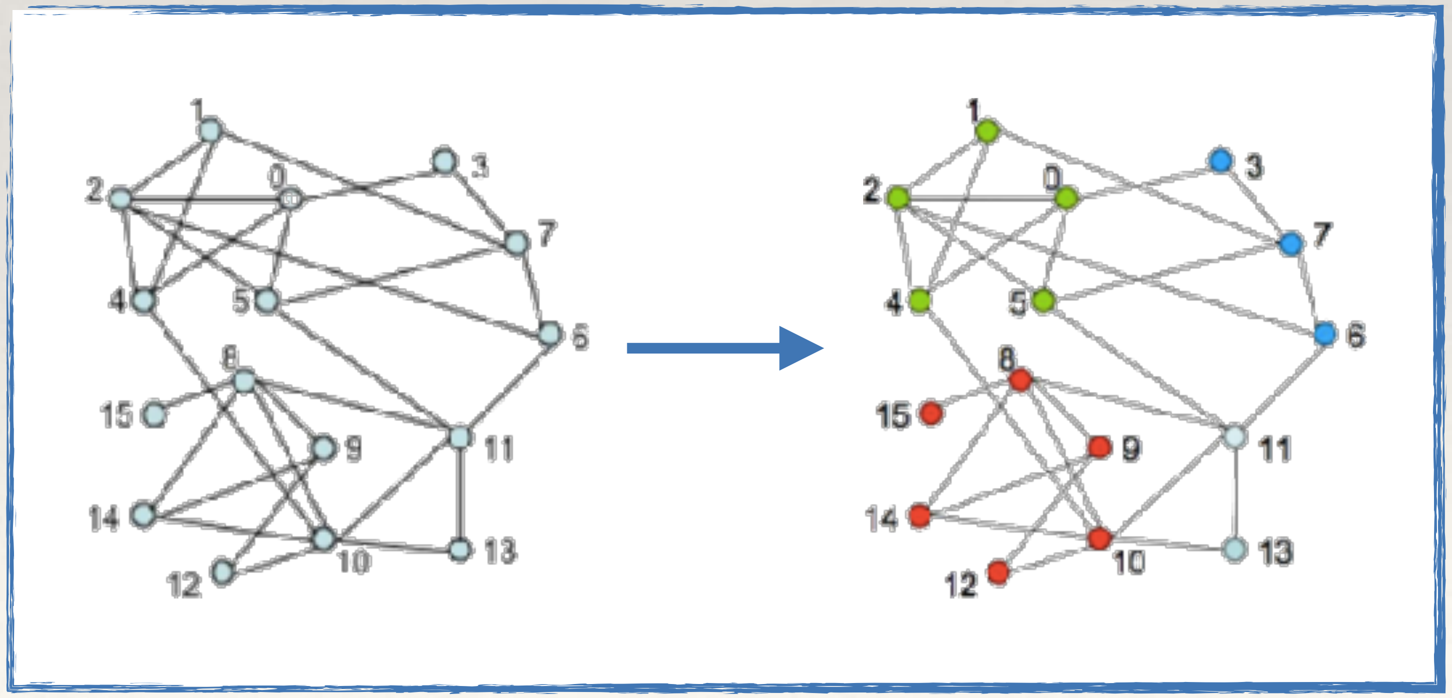

Image from [Blondel et al. 2008]

Outline

Brief Introduction to Complex Network Analysis

Persistence-based Network Analysis

Persistence-based Network Analysis

Several Application based on Persistent Homology:

- *Sensor* Networks [De Silva 2013]
- *Brain* Networks [Lee et al. 2012]
- *Collaborative/Co-occurence* Networks [Carstens et al. 2013; Rieck et al. 2016]
- *Geolocalized* Networks [Fellegara et al. 2016]
- …

Simplicial Complex Representation:

A network is represented through:

- Simplicial complex *Flag(G)* induced by G
  - *simplices of Flag(G)* ⟷ *cliques of G*

Persistence-based Network Analysis

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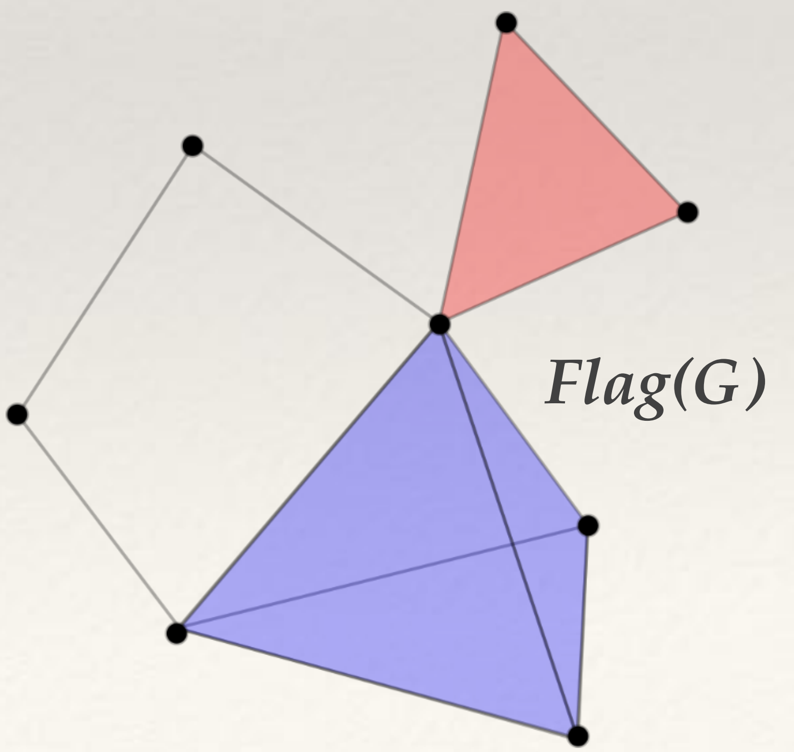
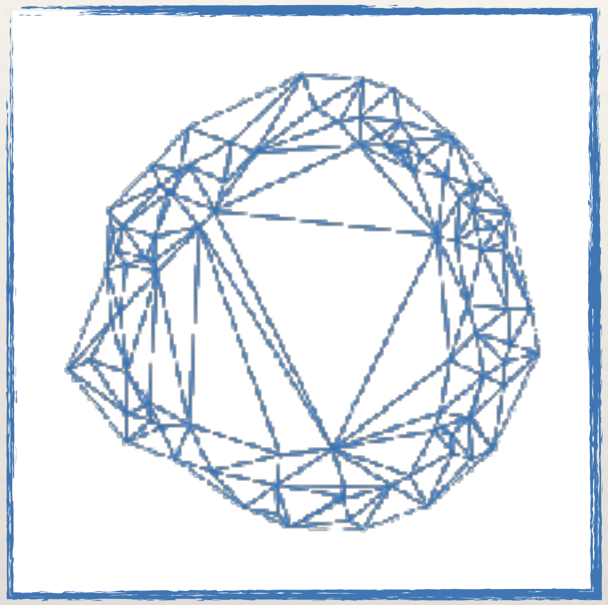
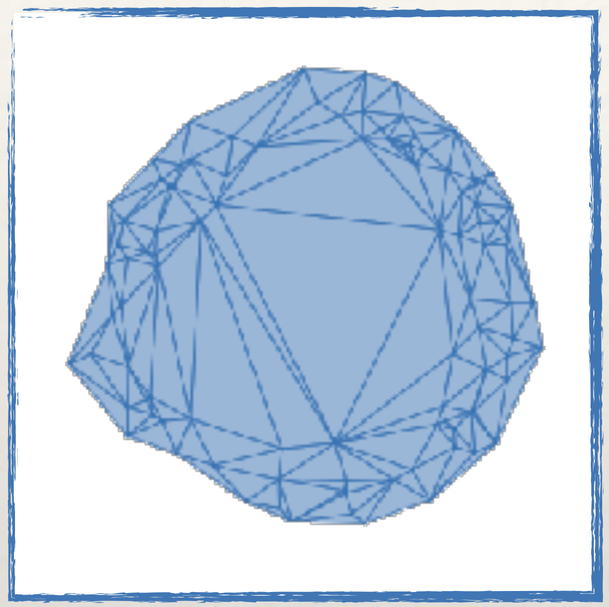
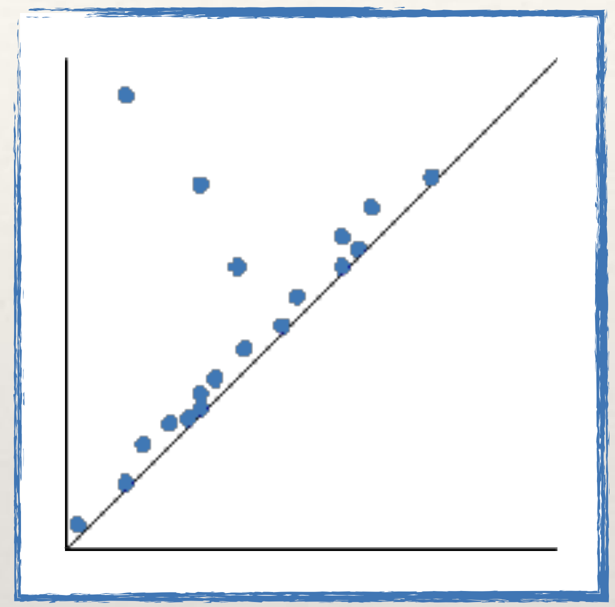


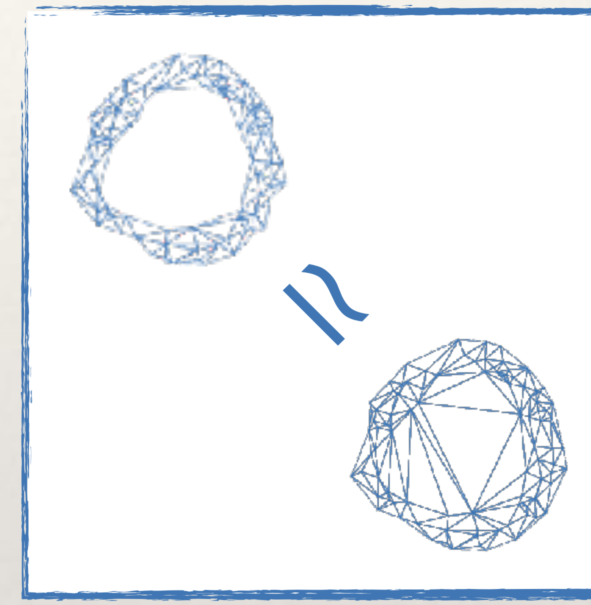
Persistence-based Network Analysis

A Common Pipeline in TDA:









Complex Construction

PH_k module

Distance Computation

Topological Summaries have proven to be particularly effective to **distinguish shapes**

but

It's still hard to give a **meaningful interpretation** of what homological cycles represent

Outline

Brief Introduction to
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Persistence-based
Network Analysis

Thank you

Ulderico Fugacci

***TU Kaiserslautern**, Dept. of Computer Science*