“Persistent Homology” Summer School - Rabat

Persistent Homology Computation and Discrete Morse Theory

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Outline

Persistent Homology

Computation

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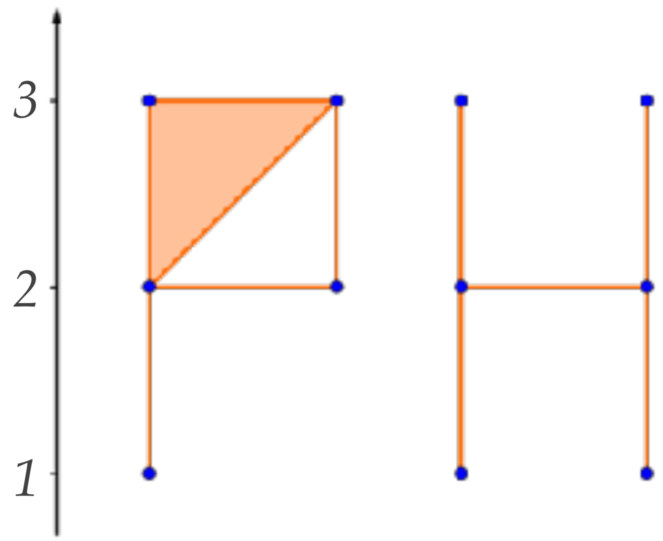
Persistent Homology
Computation

Discrete Morse Theory and
Persistent Homology

Computing Persistent Homology

Standard Algorithm: [Edelsbrunner et al. 2002; Zomorodian, Carlsson 2005]

From:



To:

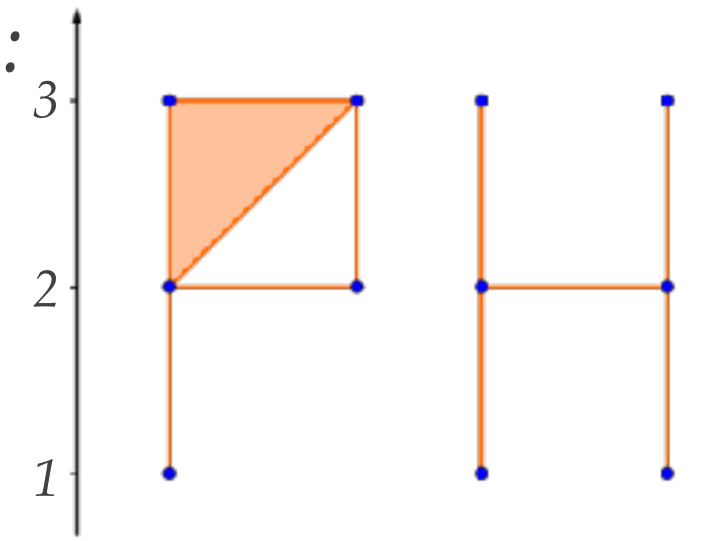
H_0 $[1, 2]$, $[1, \infty)$, $[1, \infty)$

H_1 $[3, \infty)$

Computing Persistent Homology

Standard Algorithm: [Edelsbrunner et al. 2002; Zomorodian, Carlsson 2005]

From:



To:

H_0 $[1, 2]$, $[1, \infty)$, $[1, \infty)$

H_1 $[3, \infty)$

Compute a ***reduced boundary matrix*** for Σ^f
from which easily read the persistence pairs

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
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10																							
11																							
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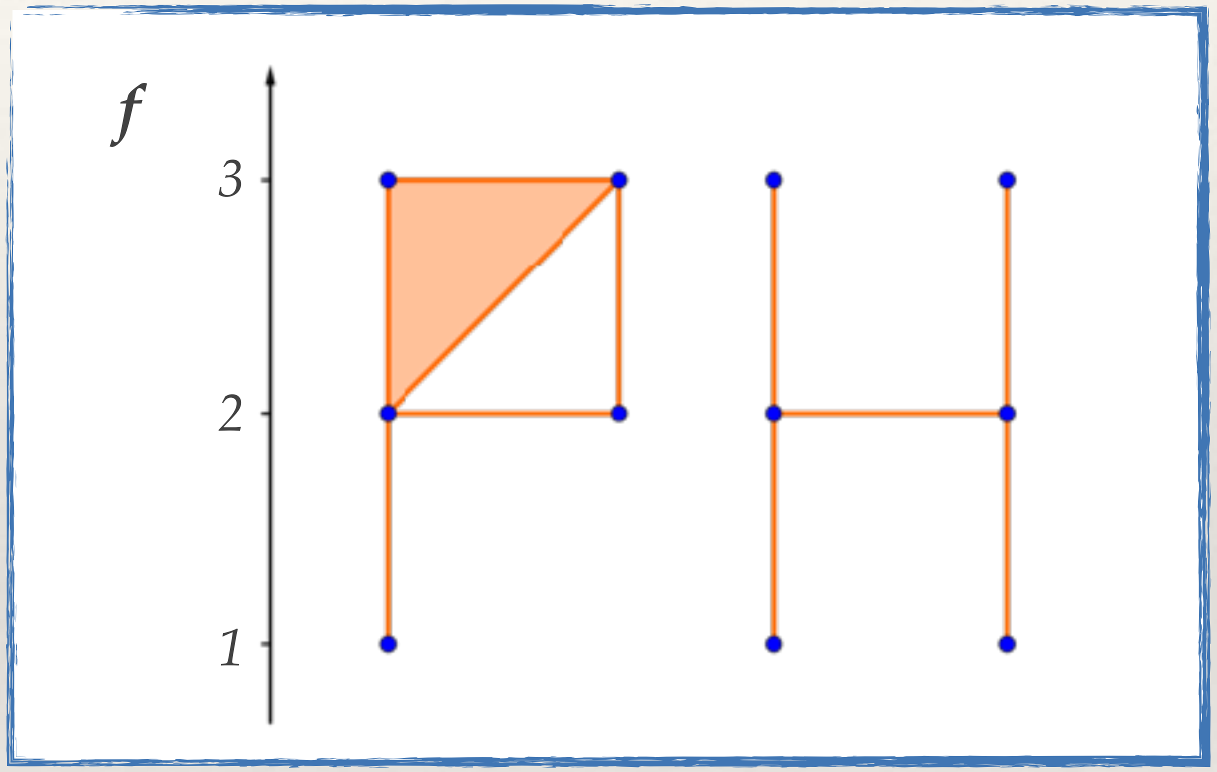
14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

Computing Persistent Homology

Given a filtered simplicial complex, let us consider its ***filtering function f*:**

$$f(\sigma) := min\{ p \mid \sigma \in \Sigma^p \}$$



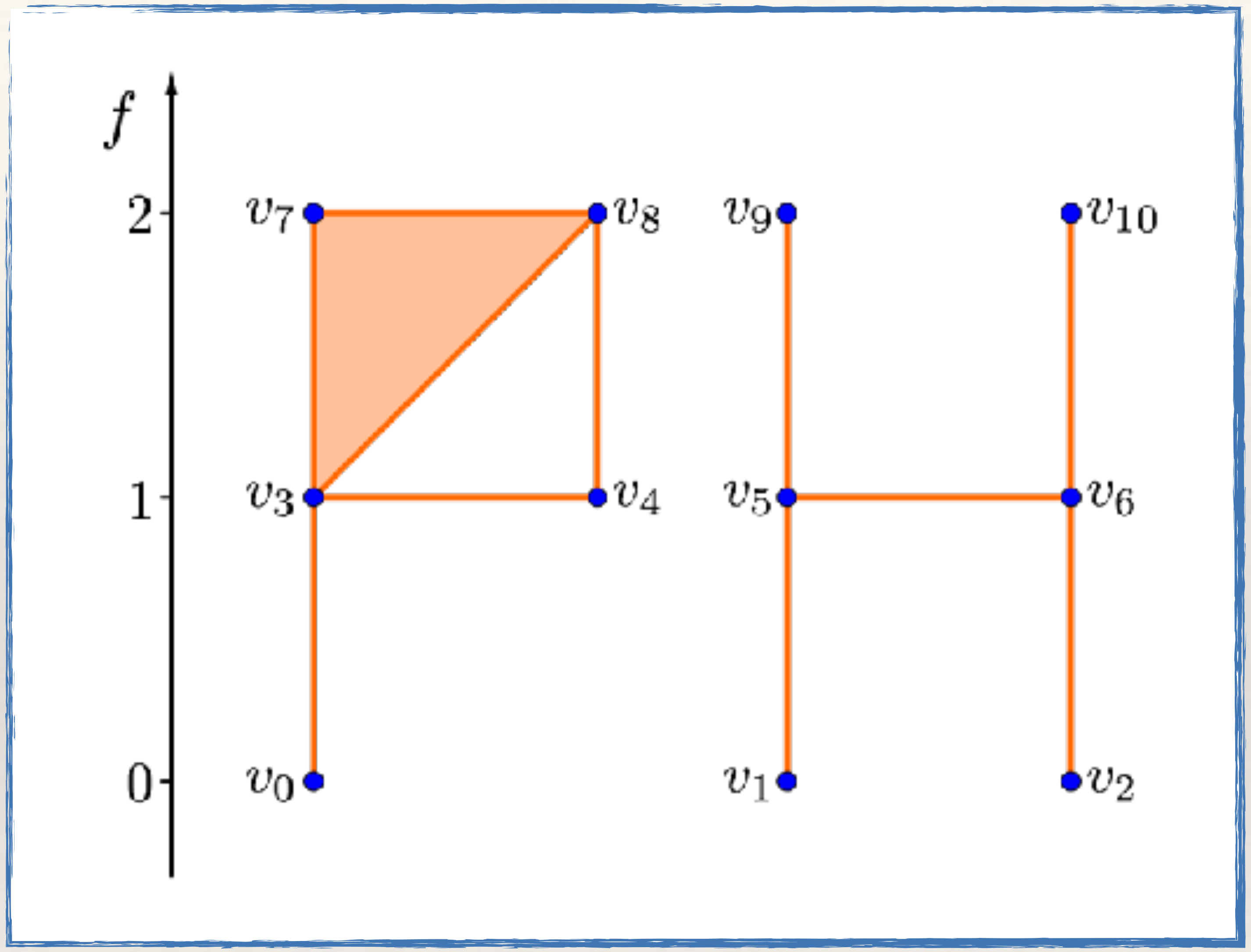
Conversely, $\Sigma^p := \{ \sigma \in \Sigma \mid f(\sigma) \leq p \}$

Total Ordering on Σ^f:

A sequence $\sigma_1, \sigma_2, \dots, \sigma_n$ of the simplices of Σ such that:

- if $f(\sigma_i) < f(\sigma_j)$, then $i < j$
- if σ_i is a proper face of σ_j, then $i < j$

Computing Persistent Homology



A possible choice:

Set $\sigma < \sigma'$ if:

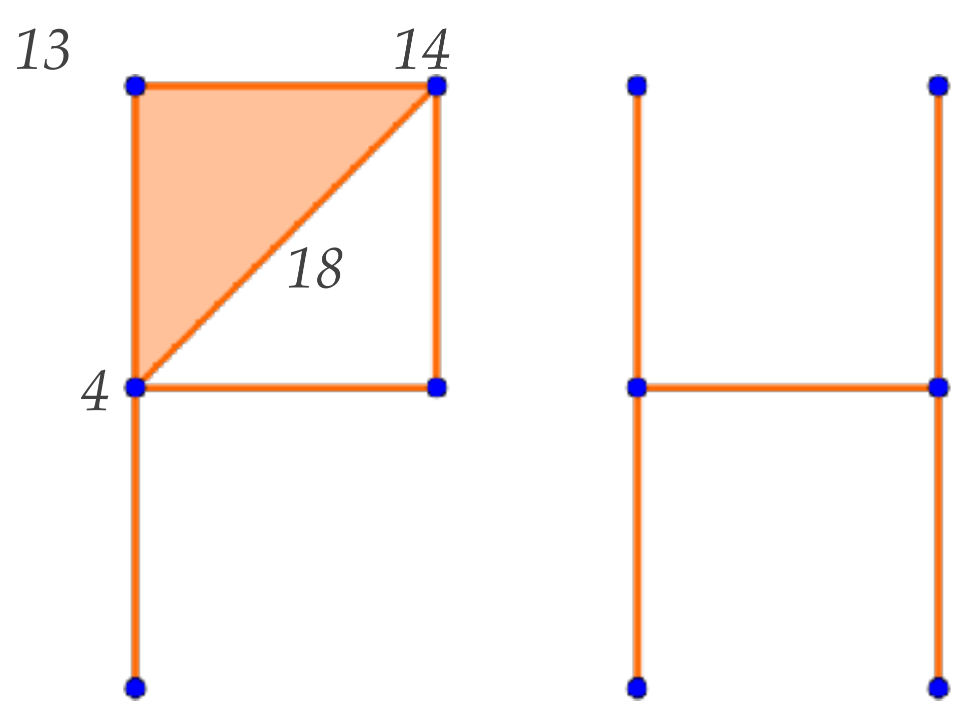
- if $f(\sigma) < f(\sigma')$
- if $f(\sigma) = f(\sigma')$ and $dim(\sigma) < dim(\sigma')$
- if $f(\sigma) = f(\sigma')$ and $dim(\sigma) = dim(\sigma')$ and σ precedes σ' with respect to the *lexicographic order* of their vertices

Computing Persistent Homology

Boundary Matrix:

A square matrix ***M*** of size *n* x *n* defined by

$$M_{i,j} := \begin{cases} 1 & if\ \sigma_i\ is\ a\ face\ of\ \sigma_j\ s.t.\ dim(\sigma_i) = dim(\sigma_j) - 1 \\ 0 & otherwise \end{cases}$$



E.g.,

- $M_{4,18} = 1$
- $M_{14,18} = 1$
- $M_{13,18} = 0$

Computing Persistent Homology

Reduced Matrix:

Given a non-null column j of a boundary matrix M,

$$low(j) := max \{ i \mid M_{i,j} \neq 0 \}$$

A matrix R is called ***reduced*** if, for each pair of nun-null columns j_1, j_2,

$$low(j_1) \neq low(j_2)$$

Equivalently, if low function is ***injective*** on its domain of definition

Computing Persistent Homology

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1													
4								1			1						1	1					
5											1								1				
6									1			1								1			
7										1		1									1		
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14																		1	1			1	
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	7					13	14	14	15	16	14	22

$\boldsymbol{low(10) = 7 = low(12)}$

$\Rightarrow$

M is **not** reduced

Computing Persistent Homology

Reduction Algorithm:

Matrix $R = M$

`for` $j = 1, \ldots, n$ `do`

  `while` $\exists\, j'' < j$ with $low(j') = low(j)$ `do`

    $R.column(j) = R.column(j) + R.column(j')$

  `endwhile`

`endfor`

`return` R

Time Complexity:

At most n^2 column additions ⇒ $O(n^3)$ in the worst case

Initialization:

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1													
4								1			1						1	1					
5											1								1				
6									1			1								1			
7										1		1									1		
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21																							
22																							1
23																							
low								4	6	7	5	7					13	14	14	15	16	14	22

Initialize ***R*** to ***M*,** where

M is the ***boundary matrix*** of Σ^f

expressed according with a ***total ordering*** of its simplices

Step 1:

$j < 12$

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1													
4								1			1						1	1					
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21																							
22																							1
23																							
low								4	6	7	5	7					13	14	14	15	16	14	22

For each $j < 12$,

there is **no** $j' < j$ such that $low(j') = low(j)$

So, **increase** j **by 1**

Step 2:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1													
4								1			1						1	1					
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21																							
22																							1
23																							
low								4	6	7	5	7					13	14	14	15	16	14	22

For ***j* = 12, *low*(12) = 7**

Step 2:

										j'		j											
$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1													
4								1			1						1	1					
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19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	7					13	14	14	15	16	14	22

For ***j* = 12, *low*(12) = 7**

column j'=10 is such that ***low*(*j'*) = *low*(*j*) = 7**

So, set

column 12 := *column 12* + *column 10*

Step 2:

j

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
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3										1		1											
4								1			1						1	1					
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20																							
21																							
22																							1
23																							
low								4	6	7	5	6					13	14	14	15	16	14	22

For $j = 12$, $low(12) = 7$

column j'=10 is such that $low(j') = low(j) = 7$

So, set

$\text{column } 12 := \text{column } 12 + \text{column } 10 \longrightarrow low(12) = 6$

Step 2:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1		1											
4								1			1						1	1					
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21																							
22																							1
23																							
low								4	6	7	5	6					13	14	14	15	16	14	22

For $j = 12$, $low(12) = 6$

Step 2:

j' j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1														
3										1		1											
4								1			1						1	1					
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7										1											1		
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19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	6					13	14	14	15	16	14	22

For ***j* = 12, *low*(12) = 6**

***column j′* = 9** is such that ***low*(*j′*) = *low*(*j*) = 6**

So, set

column 12 := *column 12* + *column 9*

Step 2:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
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17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	14	15	16	14	22

For $j = 12$, $low(12) = 6$

$column\ j' = 9$ is such that $low(j') = low(j) = 6$

So, set

$column\ 12 := column\ 12 + column\ 9 \longrightarrow low(12) = 3$

Step 2:

j

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1								1				
6									1											1			
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14																		1	1			1	
15																				1			
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17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	14	15	16	14	22

For each $j = 12$,

there is **no** $j' < j$ such that

$low(j') = low(j) = 3$

So, **increase j by 1**

Step 3:

$12 < j < 19$

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1								1				
6									1											1			
7										1											1		
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15																				1			
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17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							

low								4	6	7	5	3					13	14	14	15	16	14	22

For each ***12 < j < 19***,

there is **no** ***j' < j*** such that

low(j') = low(j)

So, **increase *j* by *1***

Step 4:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1								1				
6									1											1			
7										1											1		
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14																		1	1			1	
15																				1			
16																					1		
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18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	14	15	16	14	22

For $j = 19$, $low(19) = 14$

Step 4:

j' j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1								1				
6									1											1			
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18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	14	15	16	14	22

For $j = 19$, $low(19) = 14$

column $j' = 18$ is such that $low(j') = low(j) = 14$

So, set

column 19 := column 19 + column 18

Step 4:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1	1				
5											1								1				
6									1											1			
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17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	5	15	16	14	22

For ***j = 19, low(19) = 14***

column j'= 18 is such that ***low(j') = low(j) = 14***

So, set

column 19 := ***column 19*** + ***column 18*** $\longrightarrow$ ***low(19) = 5***

Step 4:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1	1				
5											1								1				
6									1											1			
7										1											1		
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20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	5	15	16	14	22

For ***j* = 19, *low*(19) = 5**

Step 4:

j' j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1	1				
5											1								1				
6									1											1			
7										1											1		
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20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14	5	15	16	14	22

For $j = 19$, $low(19) = 5$

column $j' = 11$ is such that $low(j') = low(j) = 5$

So, set

column 19 := *column 19* + *column 11*

Step 4:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
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6									1											1			
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19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	14	22

For ***j* = 19, *low*(19) = 5**

column j′= 11 is such that ***low*(*j′*) = *low*(*j*) = 5**

So, set

column 19 := ***column 19*** + ***column 11*** $\longrightarrow$ ***low*(19) *undefined***

Step 4:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
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13																	1					1	
14																		1				1	
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17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	14	22

For each ***j* = 19**,

there is **no** ***j'*** **<** ***j*** such that

***low*(*j'*) = *low*(*j*)**

So, **increase *j* by 1**

Step 5:

$19 < j < 22$

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
12																							
13																	1					1	
14																		1				1	
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	14	22

For each *19 < j < 22*,

there is **no** *j' < j* such that

low(j') = low(j)

So, **increase *j* by *1***

Step 6:

j

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
12																							
13																	1					1	
14																		1				1	
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	14	22

For $j = 22$, $low(22) = 14$

Step 6:

																	j'				j	
$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
12																							
13																	1					1	
14																		1				1	
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	14	22

For $j = 22$, $low(22) = 14$

column $j' = 18$ is such that $low(j') = low(j) = 14$

So, set

column 22 := *column 22* + *column 18*

Step 6:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1				1	
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
12																							
13																	1					1	
14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	13	22

For $j = 22$, $low(22) = 14$

$\qquad$ column $j' = 18$ is such that $low(j') = low(j) = 14$

So, set

$\text{column } 22 := \text{column } 22 + \text{column } 18 \longrightarrow low(22) = 13$

Step 6:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1				1	
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
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13																	1					1	
14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	13	22

For $j = 22$, $low(22) = 13$

Step 6:

																	j'					j	
$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1				1	
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
12																							
13																	1					1	
14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16	13	22

For $j = 22$, $low(22) = 13$

$\quad$ column $j' = 17$ is such that $low(j') = low(j) = 13$

So, set

column 22 := column 22 + column 17

Step 6:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
11																							
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14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

For $j = 22$, $low(22) = 13$

$\quad\quad$ column $j' = 17$ is such that $low(j') = low(j) = 13$

So, set

column 22 := column 22 + column 17 $\longrightarrow$ $low(22)$ *undefined*

Step 6:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
9																							
10																							
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14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

For each $j = 22$,

there is **no** $j' < j$ such that

$$low(j') = low(j)$$

So, **increase j by 1**

Step 7:

j

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
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14																		1					
15																				1			
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

For each $j = 23$,

there is **no** $j' < j$ such that

$low(j') = low(j) = 22$

So, **matrix R is reduced**

Output:

$i \backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
5											1												
6									1											1			
7										1											1		
8																							
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16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

The algorithm returns the above **reduced matrix** ***R***

Computing Persistent Homology

Retrieving Persistence Pairs:

- For each $i = 0, \ldots, n$,

  if there exists j such that $\boldsymbol{low(j) = i}$ $\Rightarrow$ ***[i, j]* is a pair for *R***

- Once every i has been parsed,

  if i is an **unpaired** value $\Rightarrow$ ***[i, ∞)* is a pair for *R***

From pairs of R to the **"actual" persistence pairs of Σ^f:**

$[i, j]$ corresponds to $[f(\sigma_i), f(\sigma_j)]$

$[i, \infty)$ corresponds to $[f(\sigma_i), \infty)$

(*homological degree* $= dim(\sigma_i)$)

Computing Persistent Homology

H_0

[1, ∞)

[2, ∞)

[3, 12]

[4, 8]

[5, 11]

[6, 9]

[7, 10]

[13, 17]

[14, 18]

[15, 20]

[16, 21]

H_1

[19, ∞)

[22, 23]

$i\backslash j$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1								1															
2									1			1											
3										1		1											
4								1			1						1	1					
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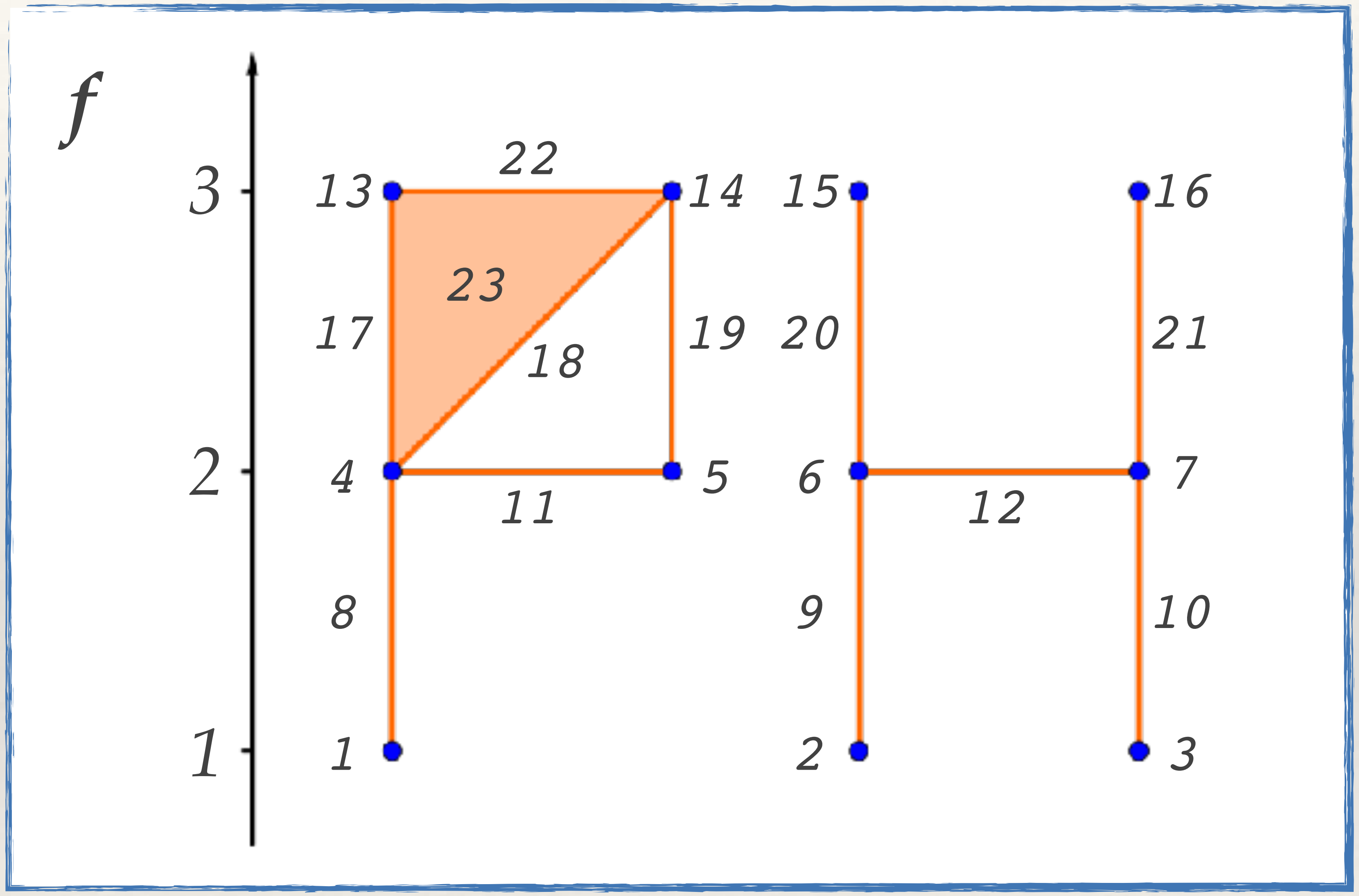
16																					1		
17																							1
18																							1
19																							
20																							
21																							
22																							1
23																							
low								4	6	7	5	3					13	14		15	16		22

Computing Persistent Homology

H_0		
[1, ∞)		[1, ∞)
[2, ∞)		[1, ∞)
[3, 12]		[1, 2]
[4, 8]		[2, 2]
[5, 11]	⇒	[2, 2]
[6, 9]		[2, 2]
[7, 10]		[2, 2]
[13, 17]		[3, 3]
[14, 18]		[3, 3]
[15, 20]		[3, 3]
[16, 21]		[3, 3]



H_1		
[19, ∞)	⇒	[3, ∞)
[22, 23]		[3, 3]

Computing Persistent Homology

Standard algorithm to compute (persistent) homology [Zomorodian, Carlsson 2005]:

- Based on a **matrix reduction**
- **Linear complexity** in practical cases
- **Quadratic complexity** in the worst case

Several different strategies:

Direct approaches

- ***Zigzag persistent homology*** [Milosavljević et al. '05]
- ***Computation with a twist*** [Chen, Kerber '11]
- ***Dual algorithm*** [De Silvia et al. '11]
- ***Output-sensitive algorithm*** [Chen, Kerber '13]
- ***Multi-field algorithm*** [Boissonnat, Maria '14]
- ***Annotation-based methods*** [Boissonnat et al. '13; Dey et al. '14]

Distributed approaches

- ***Spectral sequences*** [Edelsbrunner, Harer '08; Lipsky et al. '11]
- ***Constructive Mayer-Vietoris*** [Boltcheva et al. '11]
- ***Multicore coreductions*** [Murty et al. '13]
- ***Multicore homology*** [Lewis, Zomorodian '14]
- ***Persistent homology in chunks*** [Bauer et al. '14a]
- ***Distributed persistent computation*** [Bauer et al. '14b]

Coarsening approaches

- ***Topological operators and simplifications*** [Mrozek, Wanner '10; Dlotko, Wagner '14]
- ***Morse-based approaches*** [Robins et al. '11; Harker et al. '14; Fugacci et al. '14]

Computing Persistent Homology

Direct approaches:

- ***Zigzag persistent homology*** [Milosavljević et al. ’05]
- ***Computation with a twist*** [Chen, Kerber ’11]
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Computing Persistent Homology

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- *Spectral sequences* [Edelsbrunner, Harer ’08; Lipsky et al. ’11]
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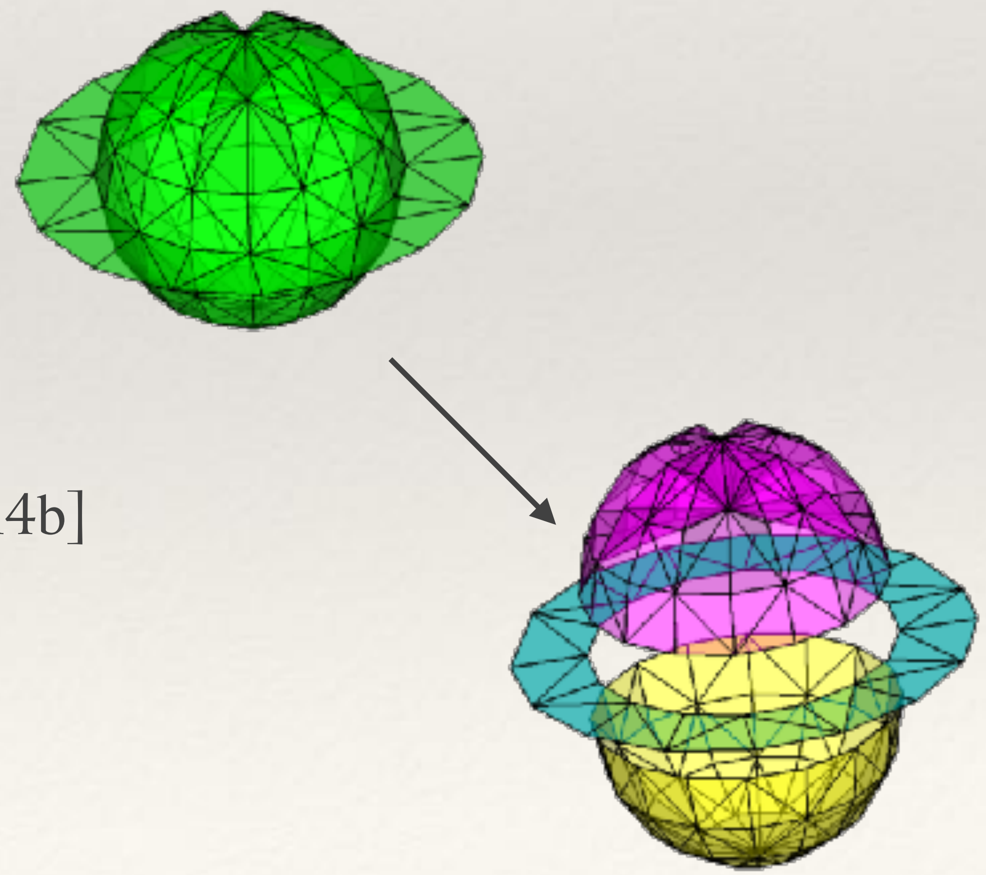


Computing Persistent Homology

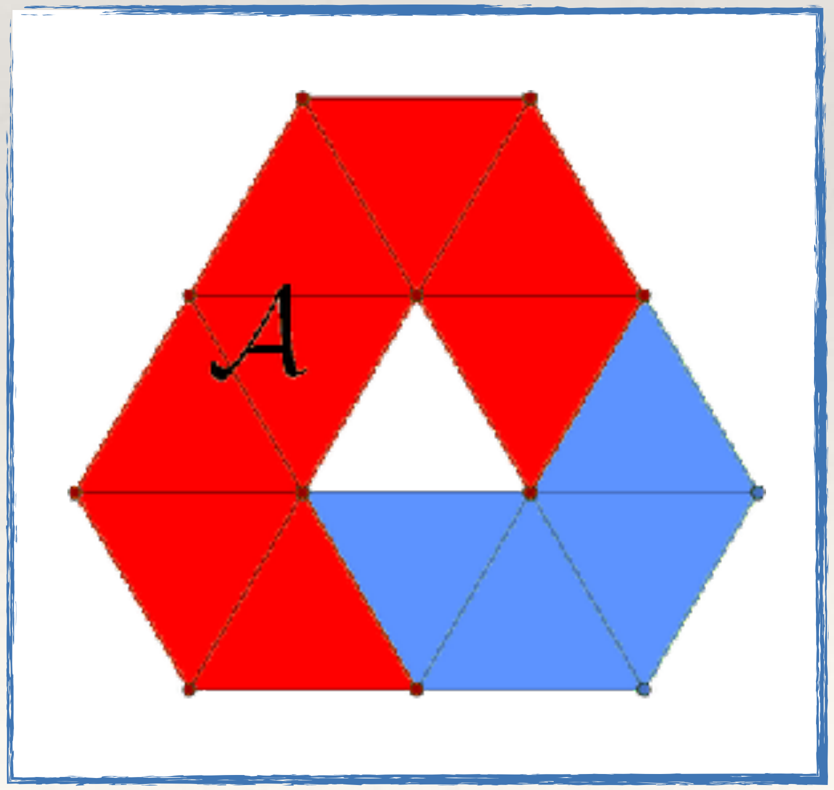
Coarsening approaches:

- *Topological operators and simplifications* [Dlotko, Wagner '14]

Computing Persistent Homology

Coarsening approaches:

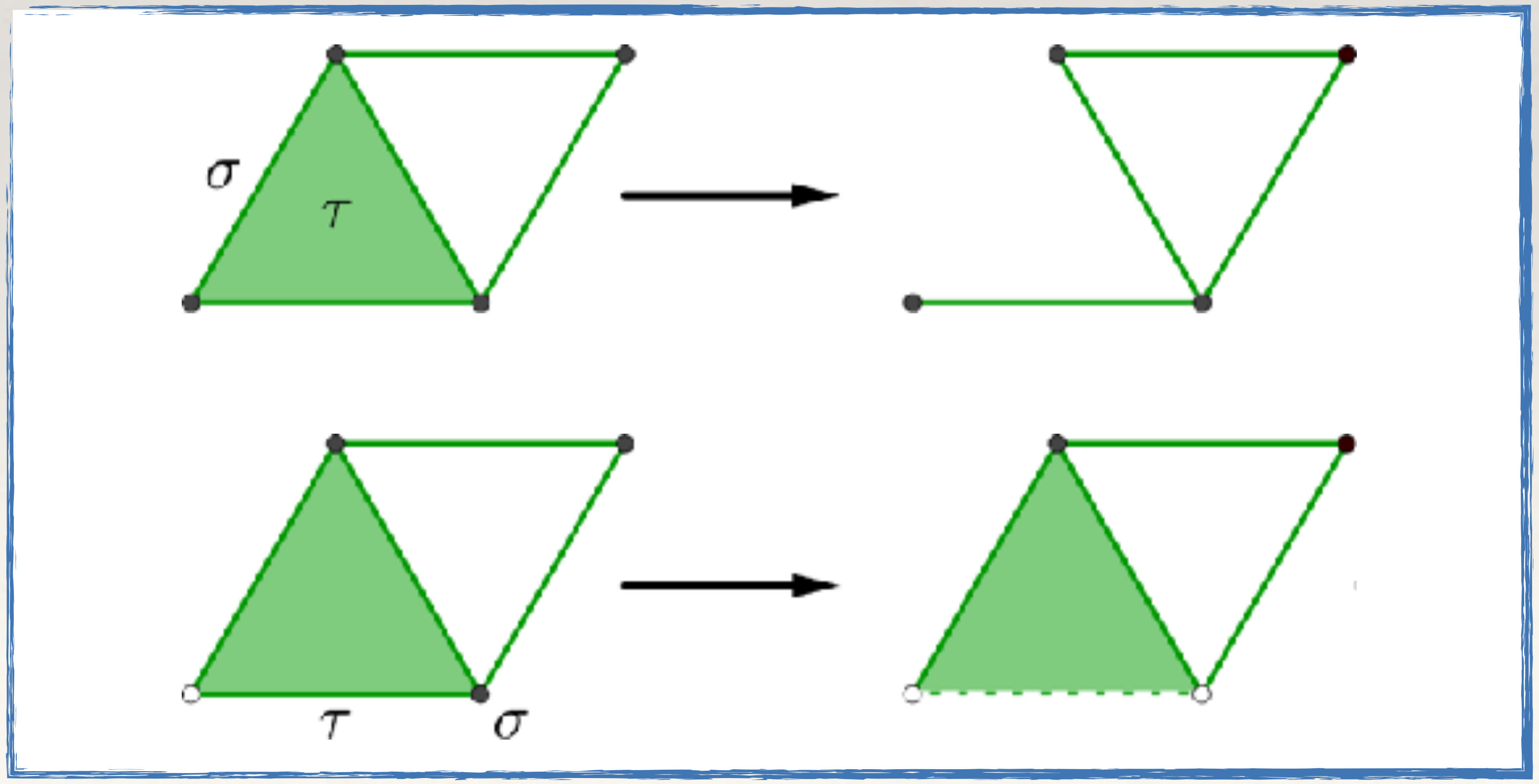
- *Topological operators and simplifications* [Dlotko, Wagner '14]
  - **Acyclic Subcomplexes** [Mrozek et al. '08]



Computing Persistent Homology

Coarsening approaches:

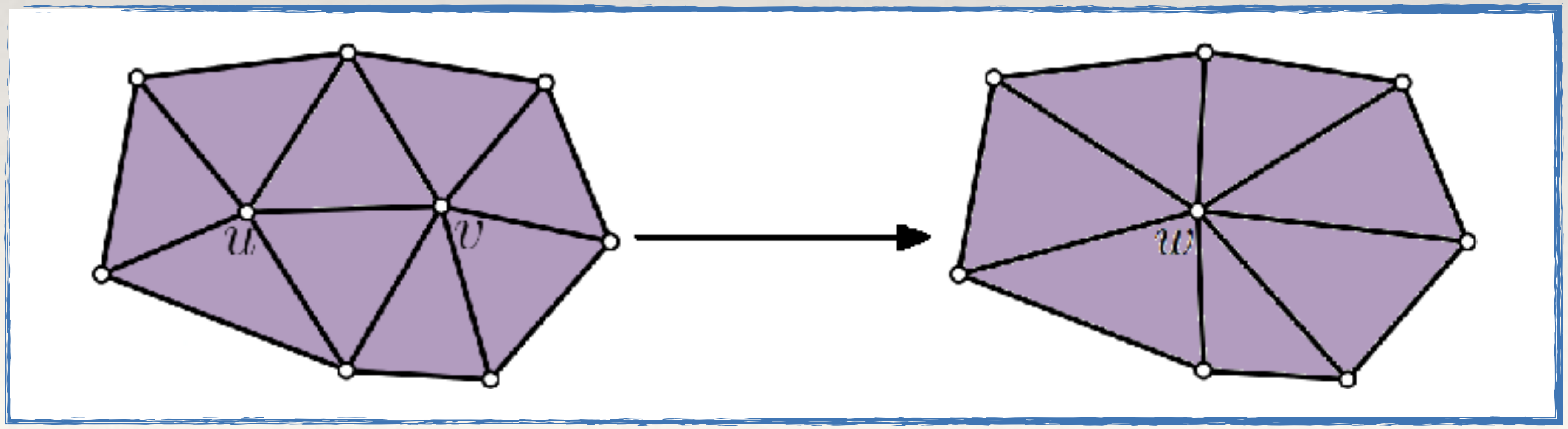
- *Topological operators and simplifications* [Dlotko, Wagner '14]
  - Acyclic Subcomplexes [Mrozek et al. '08]
  - **Reductions and Coreductions** [Mrozek et al. '10]



Computing Persistent Homology

Coarsening approaches:

- *Topological operators and simplifications* [Dlotko, Wagner '14]
  - Acyclic Subcomplexes [Mrozek et al. '08]
  - Reductions and Coreductions [Mrozek et al. '10]
  - **Edge Contractions** [Attali et al. '11]



Computing Persistent Homology

Coarsening approaches:

- ***Topological operators and simplifications*** [Dlotko, Wagner ’14]
  - Acyclic Subcomplexes [Mrozek et al. ‘08]
  - Reductions and Coreductions [Mrozek et al. ‘10]
  - Edge Contractions [Attali et al. ’11]
- ***Morse-based approaches*** [Robins et al. ’11; Harker et al. ’14; Fugacci et al. ’14]

Outline

Persistent Homology Computation

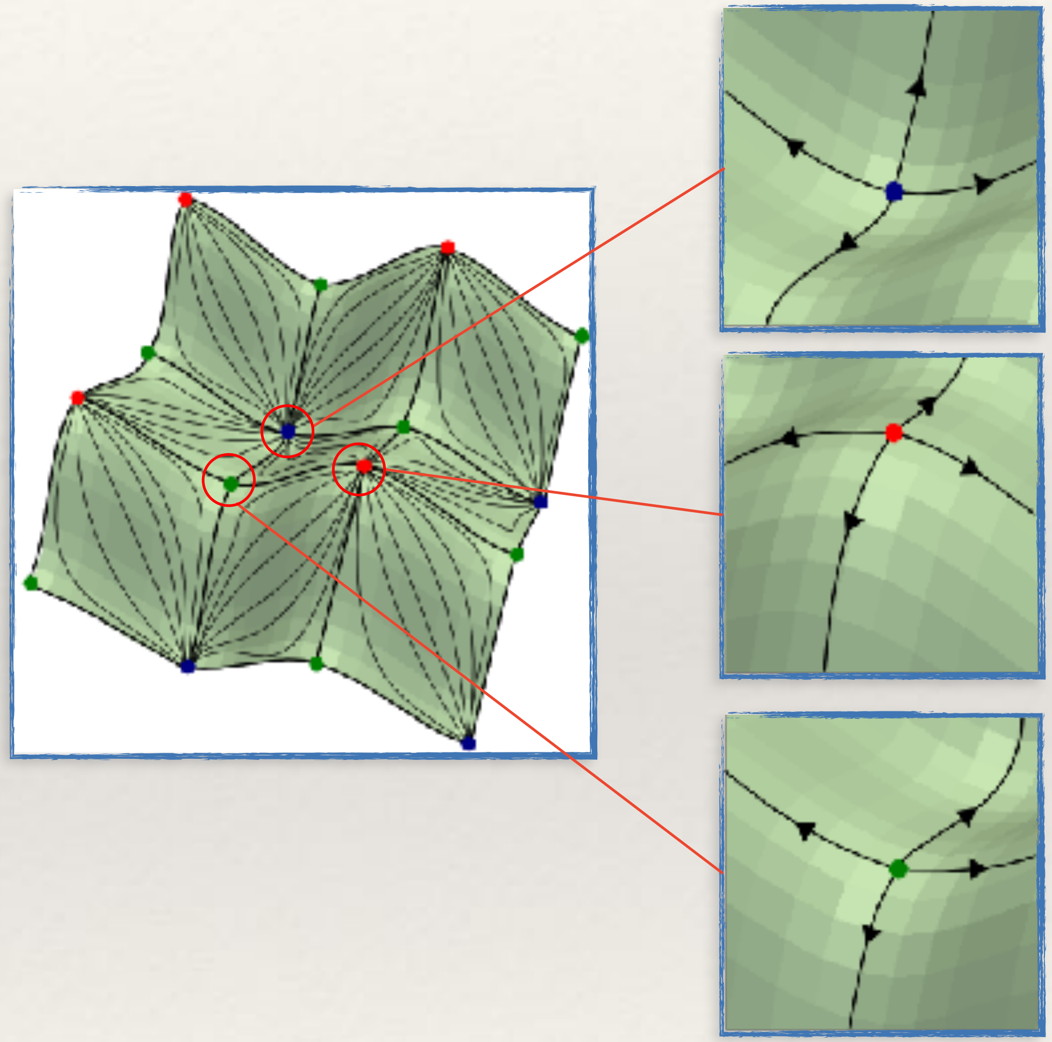
Discrete Morse Theory and Persistent Homology

Morse Theory [Milnor 1963, Matsumoto 2002]

- Topological tool for efficiently analyzing a **shape** by studying the behavior of a smooth **scalar function** *f* defined on it

- Relates the critical points of a smooth scalar function on a shape with their **regions of influence**

- Analysis of scalar fields requires extracting **morphological features** (e.g., critical points, integral lines and surfaces)

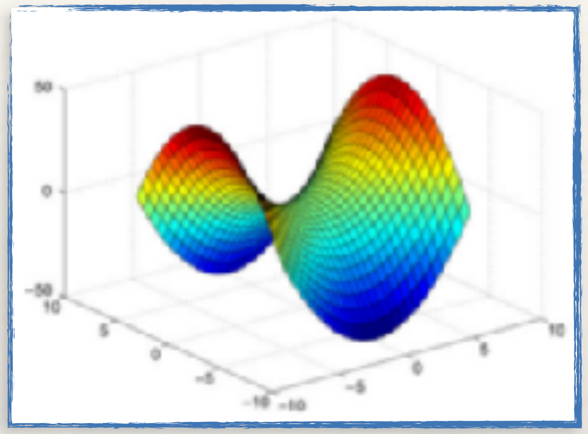


Morse Theory

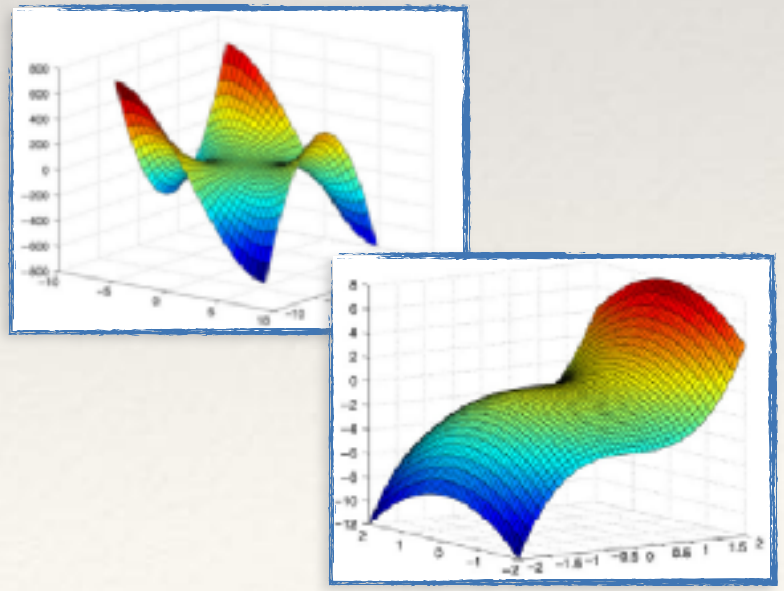
Let f be a real-valued C^2-function defined on a d-dimensional manifold M

- **Critical point** of f: any point on M in which the gradient of f vanishes
- Critical points can be **degenerate** or **non-degenerate**
  - A critical point p is degenerate iff the determinant the Hessian matrix H of the second order derivatives of function f is null

Non-degenerate critical point



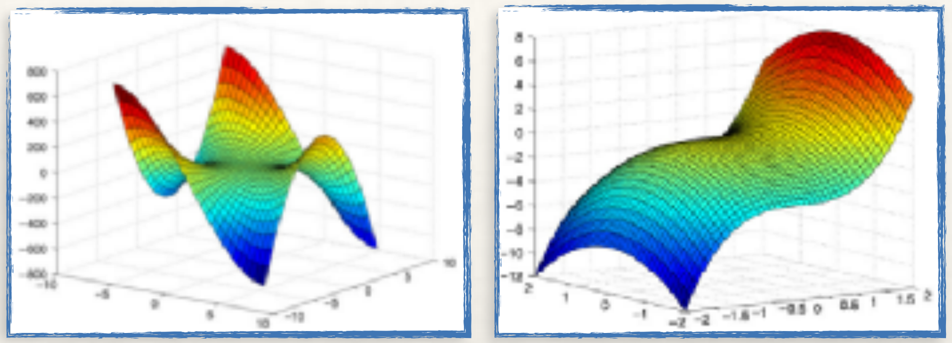
Degenerate critical points (monkey saddle and flat saddle)



Function f is a **Morse function** if and only if ***all its critical points are non-degenerate***

Morse Theory

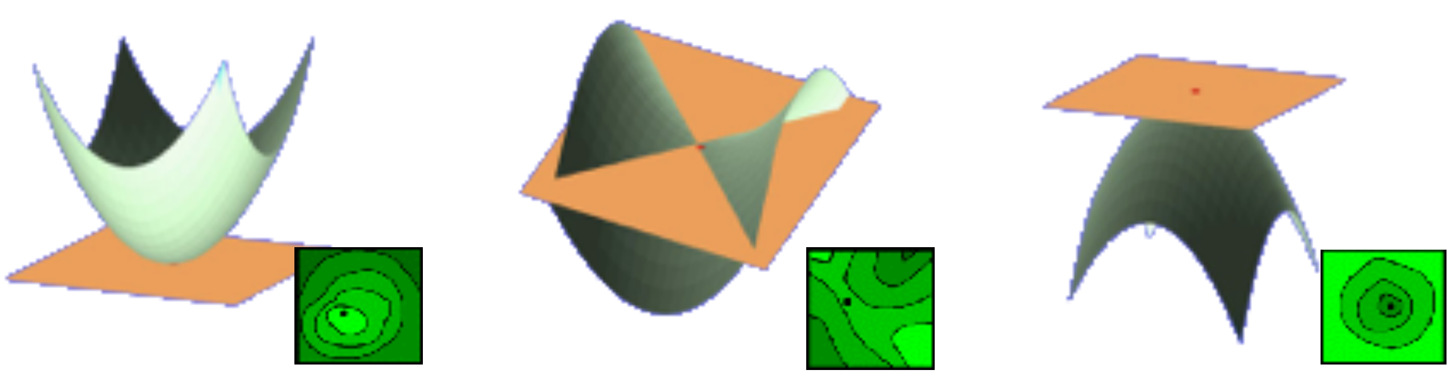
Critical points of a Morse function are ***isolated***



Examples of **non-Morse** functions

- A d-dimensional Morse function f has $d+1$ types of critical points, called ***i*-saddles** (i is the **index** of the critical point)
  - For $d = 2$: ***minima***, ***saddles*** and ***maxima***
  - For $d = 3$: ***minima***, ***1-saddles***, ***2- saddles*** and ***maxima***

Critical points of a 2D function

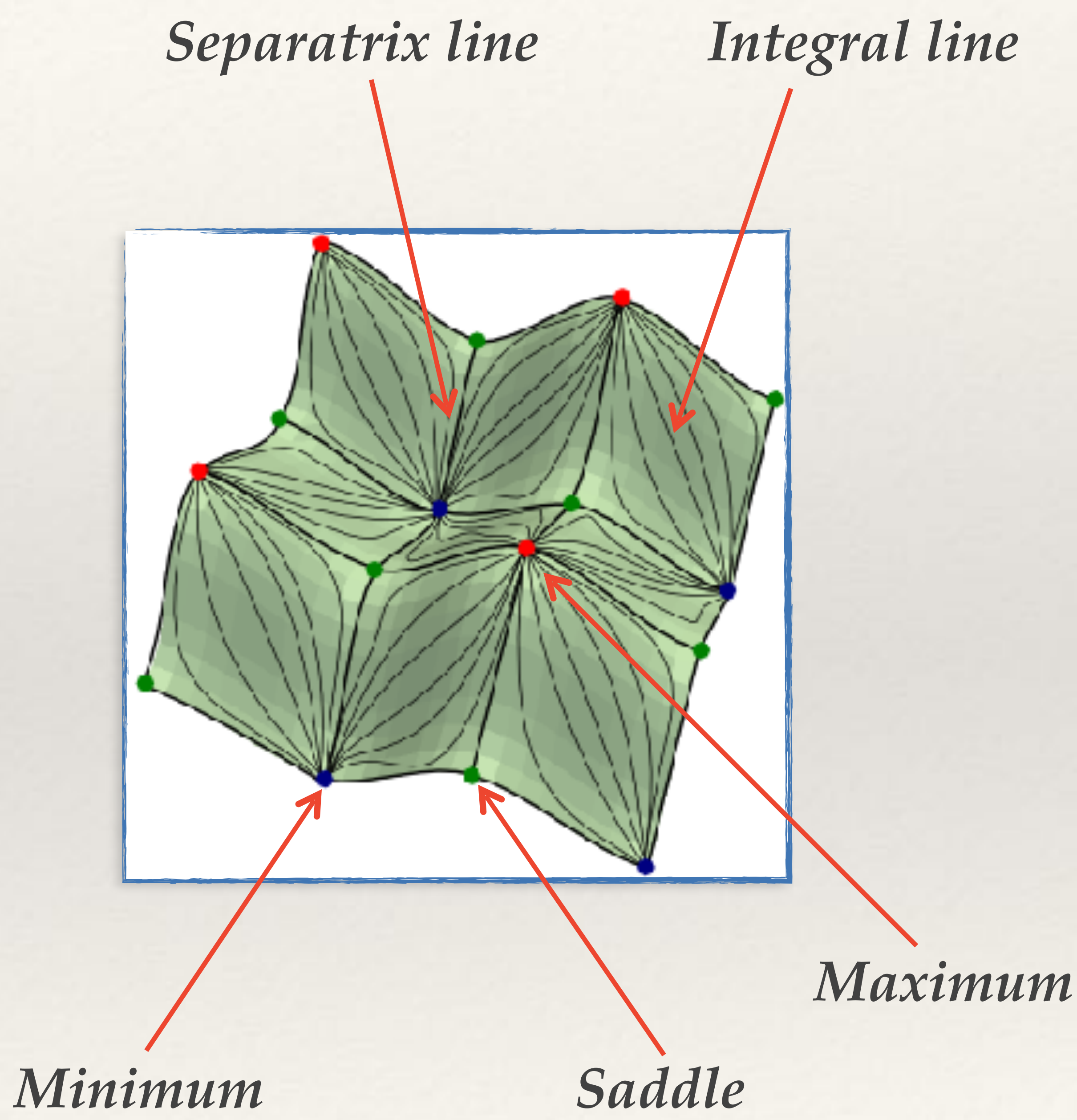


Morse Theory

- An **integral line** of a smooth function f is a maximal path which is everywhere tangent to the gradient vector field of f

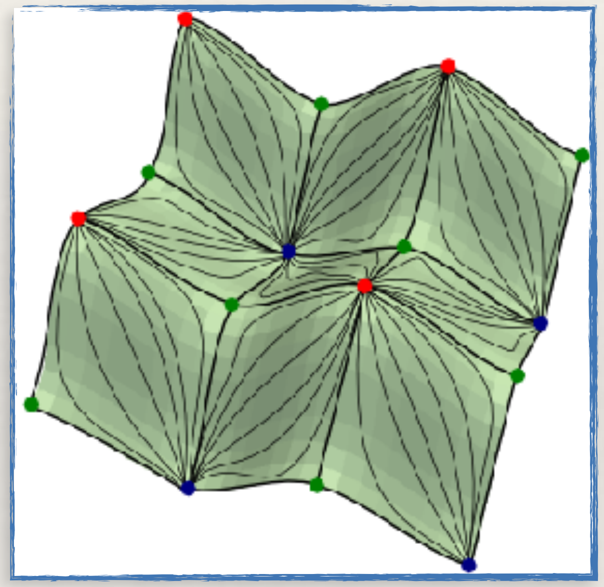
- Integral lines **start** and **end** at the critical points of f

- Integral lines that connect critical points of consecutive index are called **separatrix lines**



Morse Theory

- Integral lines that converge to a critical point p of index i form an i-cell called the **descending cell** of p
  - Descending cell of a maximum: 2-cell
  - Descending cell of a saddle: 1-cell
  - Descending cell of a minimum: 0-cell



Descending Morse Complex:

Collection of the descending cells of all critical points of function f

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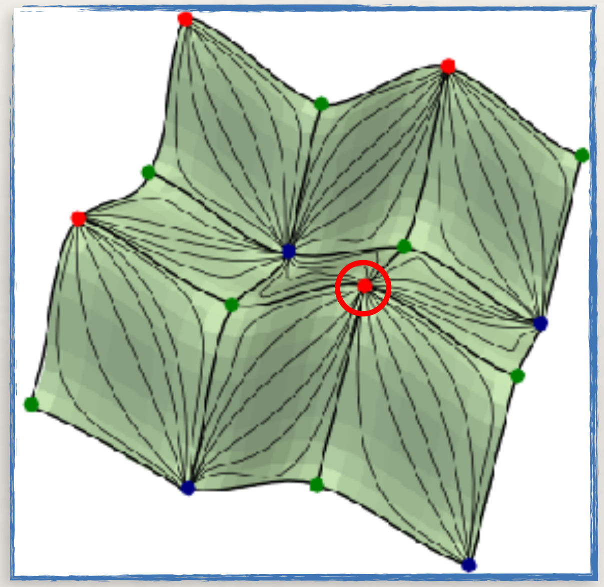

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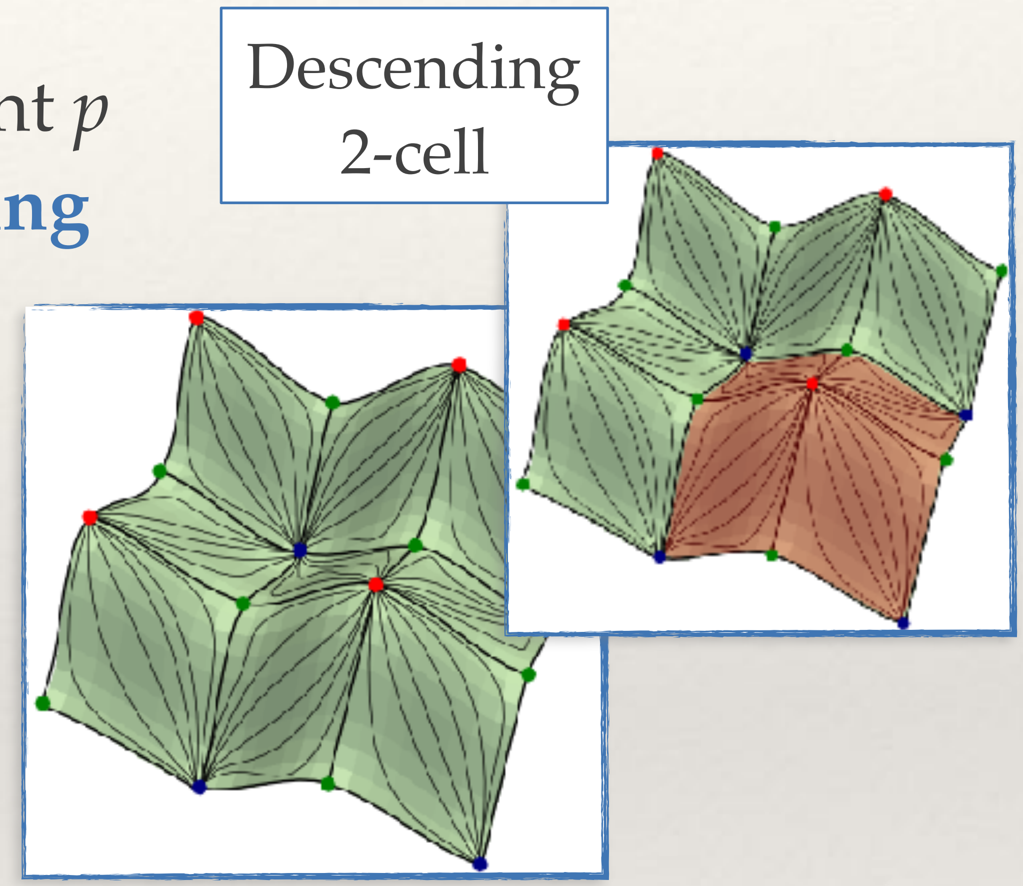

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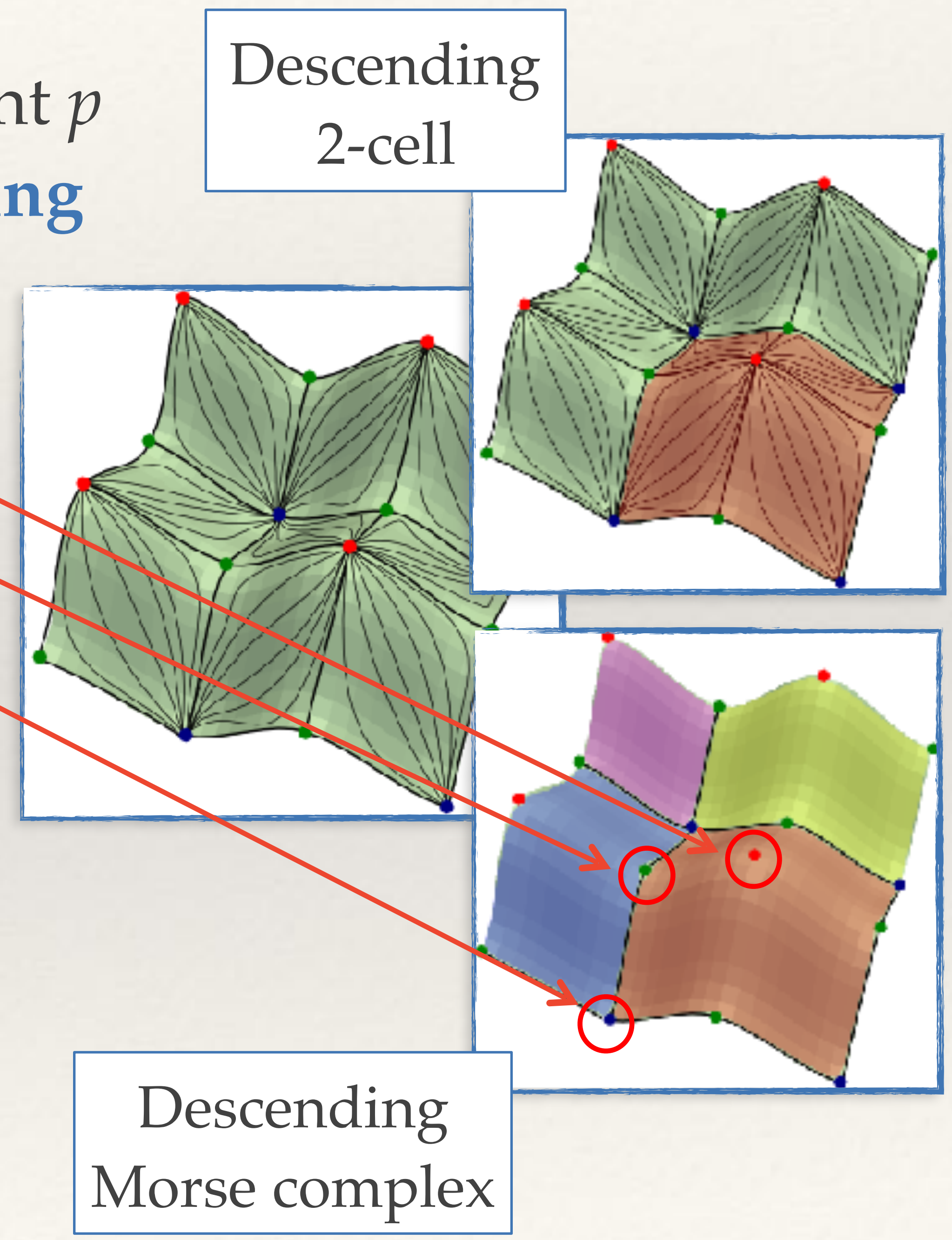
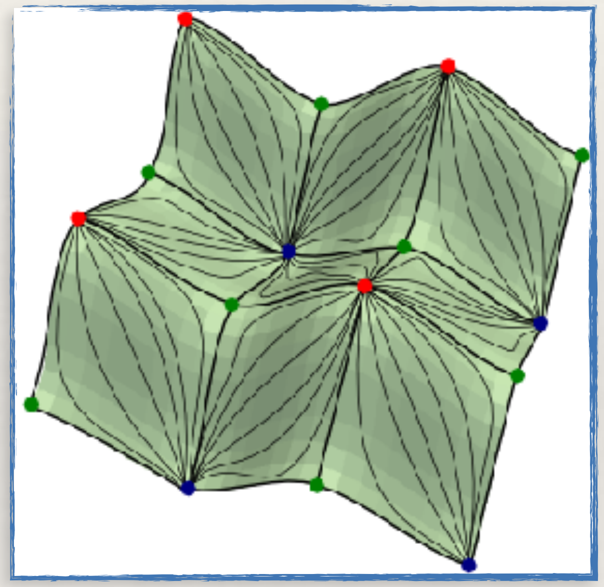


Morse Theory

- Integral lines that originate at a critical point p of index i form a $(d\text{-}i)$-cell called the **ascending cell** of p
  - Ascending cell of a minimum: 2-cell
  - Ascending cell of a saddle: 1-cell
  - Ascending cell of a maximum: 0-cell



Ascending Morse Complex:

Collection of the ascending cells of all critical points of function f

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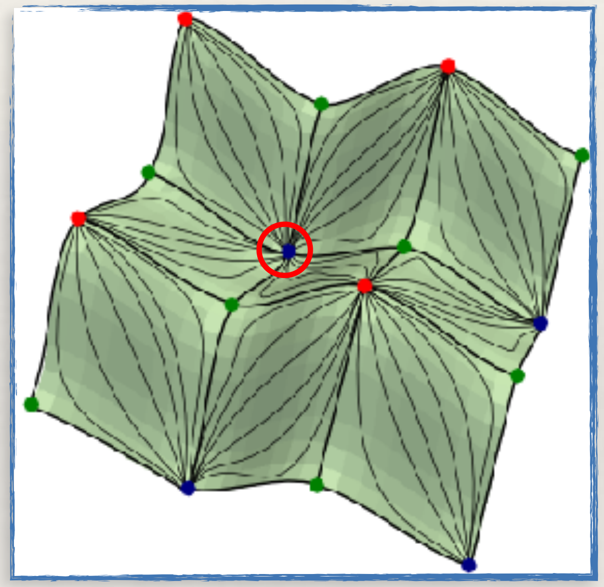


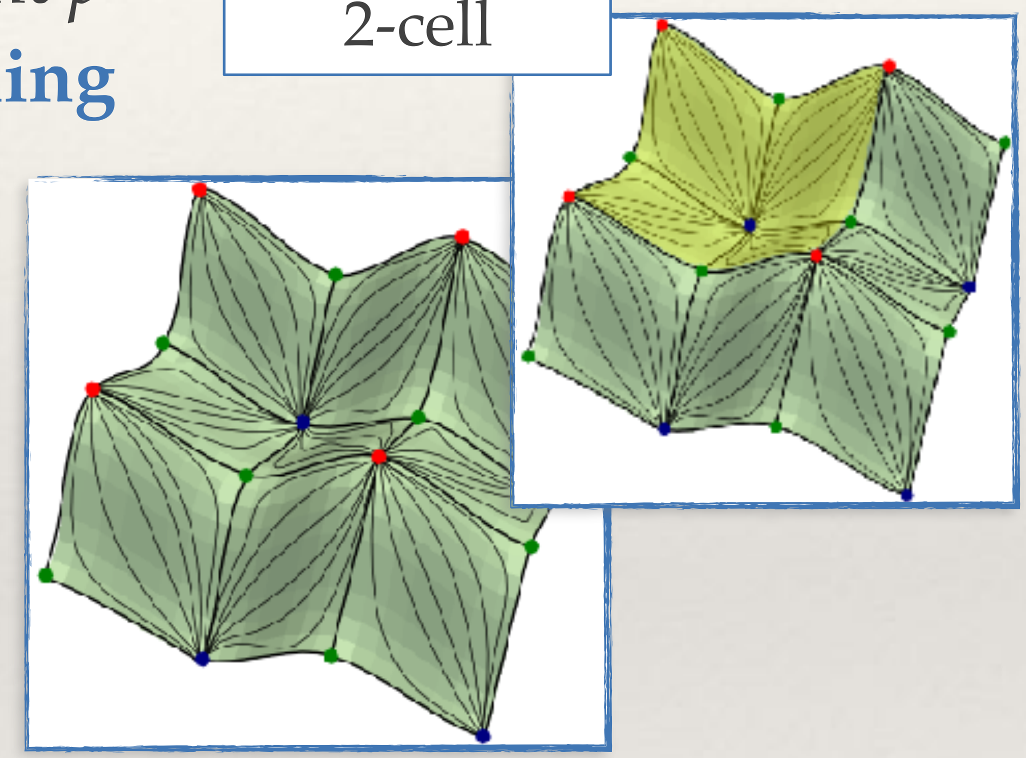
Ascending Morse Complex:

Collection of the ascending cells of all critical points of function f

Morse Theory

- Integral lines that originate at a critical point p of index i form a $(d\text{-}i)$-cell called the **ascending cell** of p
    - Ascending cell of a minimum: 2-cell
    - Ascending cell of a saddle: 1-cell
    - Ascending cell of a maximum: 0-cell

Ascending 2-cell



Ascending Morse Complex:

Collection of the ascending cells of all critical points of function f

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- Integral lines that originate at a critical point p of index i form a $(d\text{-}i)$-cell called the **ascending cell** of p
  - Ascending cell of a minimum: 2-cell
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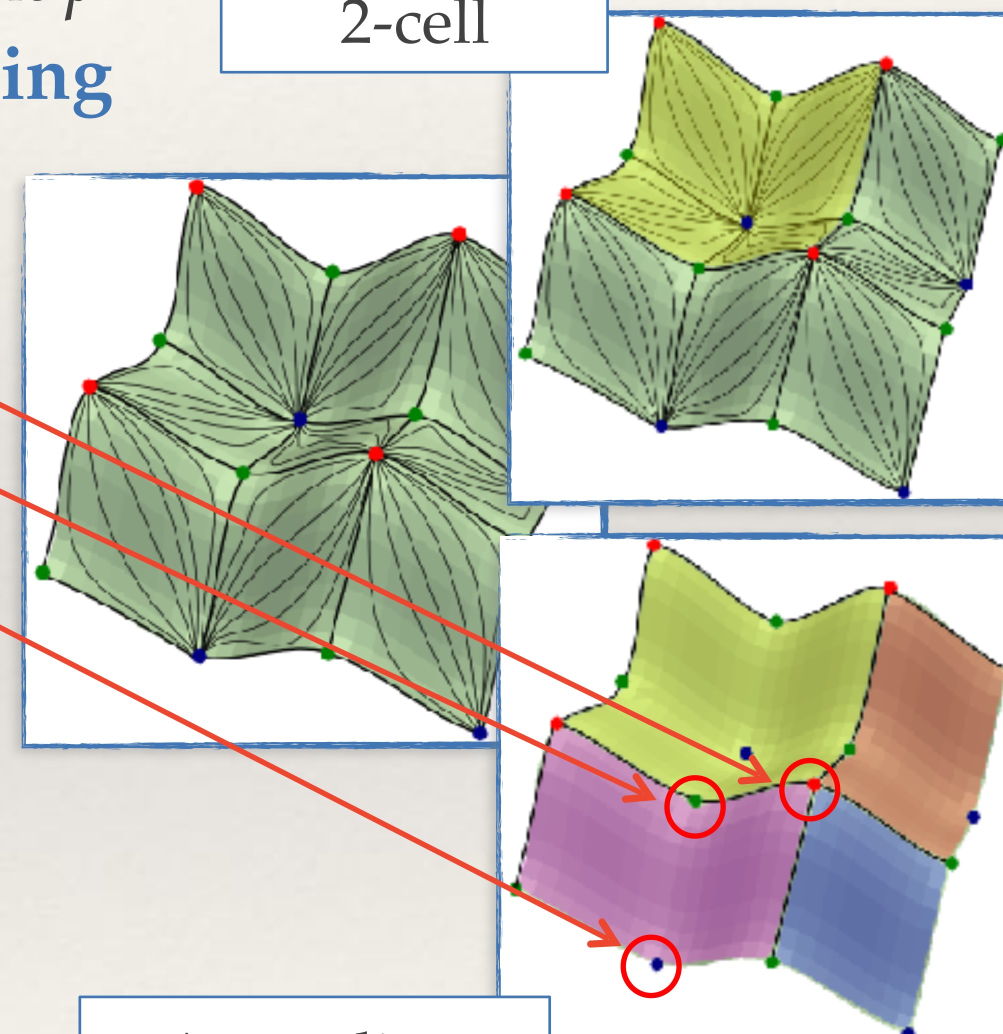


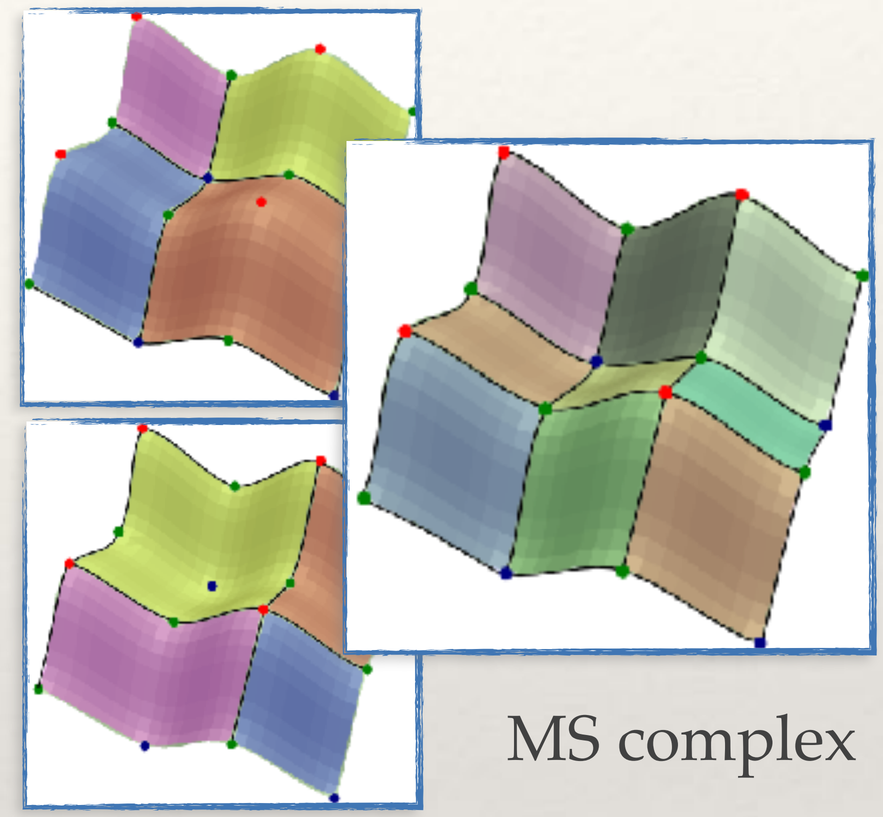
Ascending Morse complex

Ascending Morse Complex:

Collection of the ascending cells of all critical points of function f

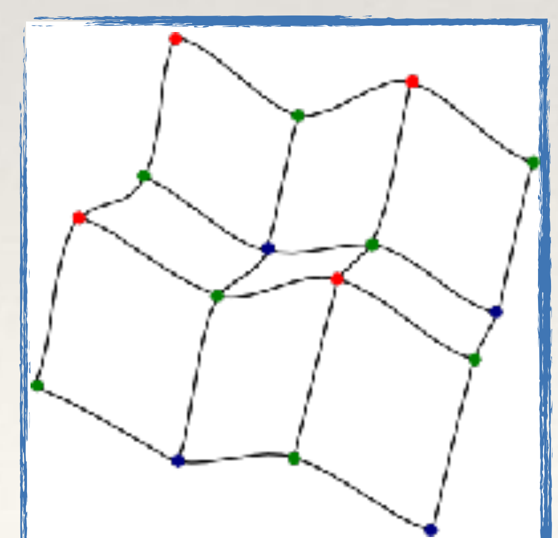
Morse Theory

- Function f is a **Morse-Smale** function if its ascending and descending Morse cells intersect transversally



MS complex

- **Morse-Smale (MS) complex** is the complex obtained from the mutual intersection of all the ascending and descending cells

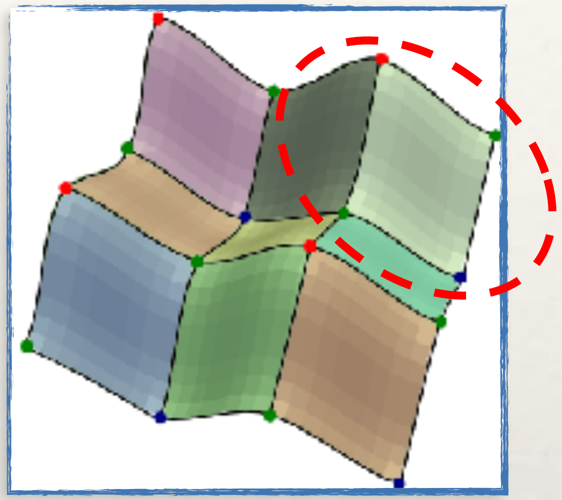


MS
1-skeleton

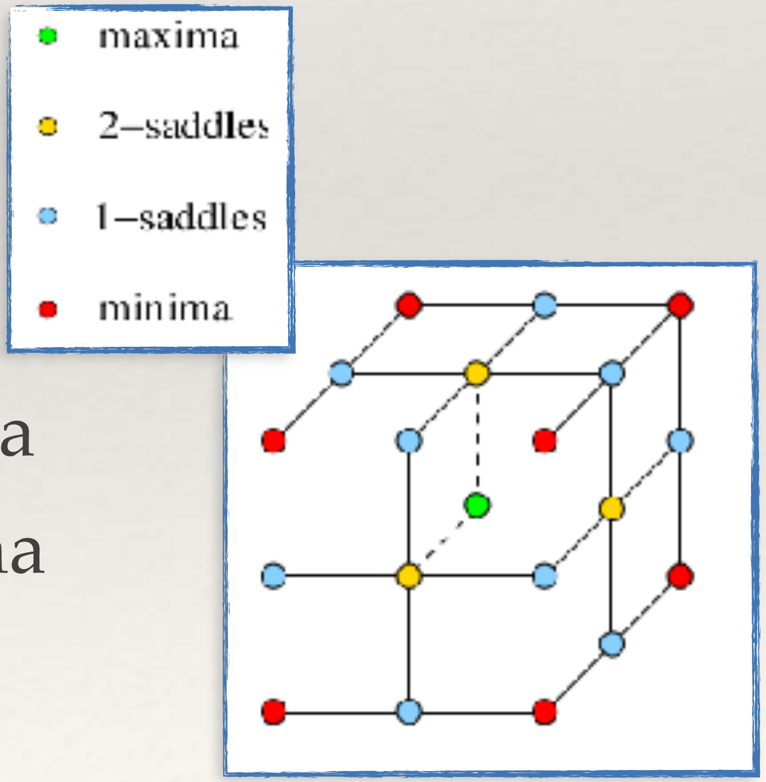
Morse Theory

- In a 2D Morse-Smale complex:
  - a 2-cell is a **quadrilateral** bounded by the sequence

    maximum – saddle – minimum – saddle



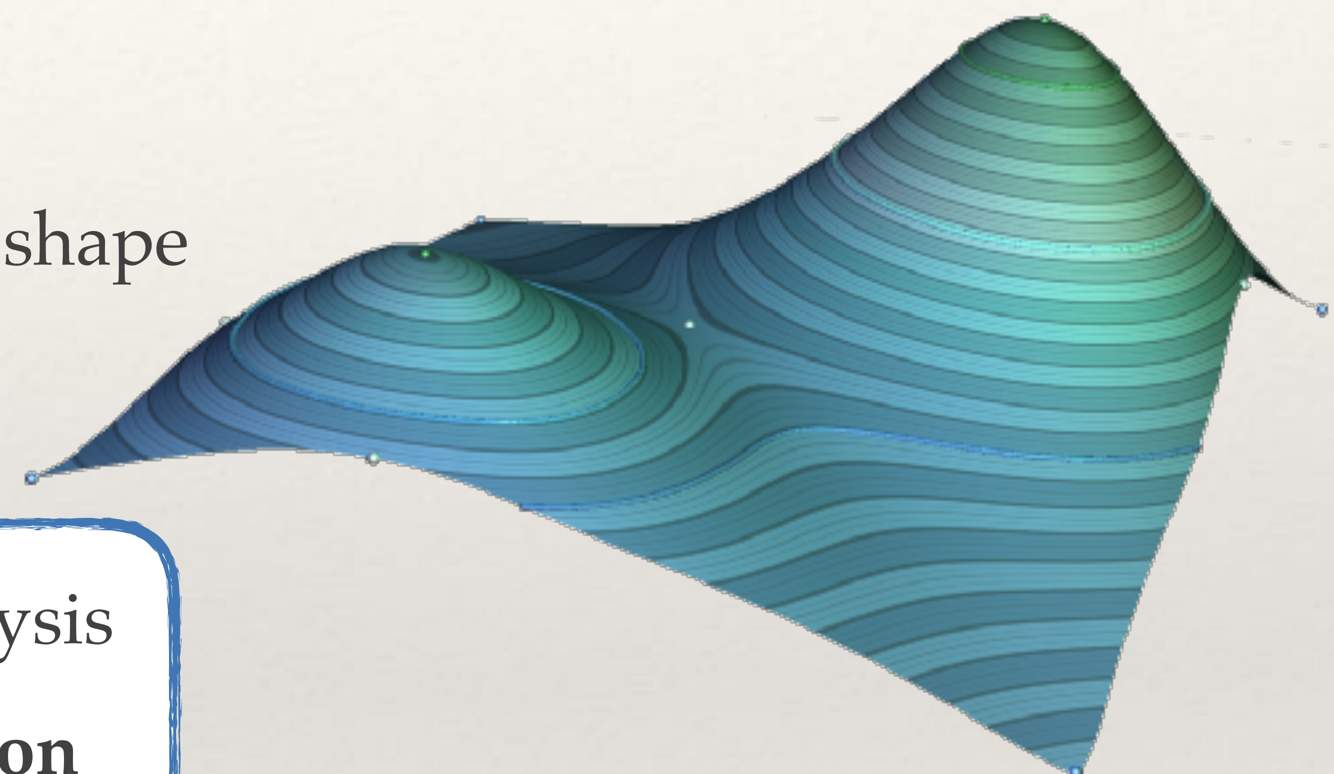
- In a 3D Morse-Smale complex:
  - each 1-saddle is connected to exactly two minima
  - each 2-saddle is connected to exactly two maxima



Discrete Morse Theory

Morse Theory:

Enables to **analyze** the **topology** of a shape by **studying functions** defined on it



Useful for
- **homological** analysis
- shape **segmentation**

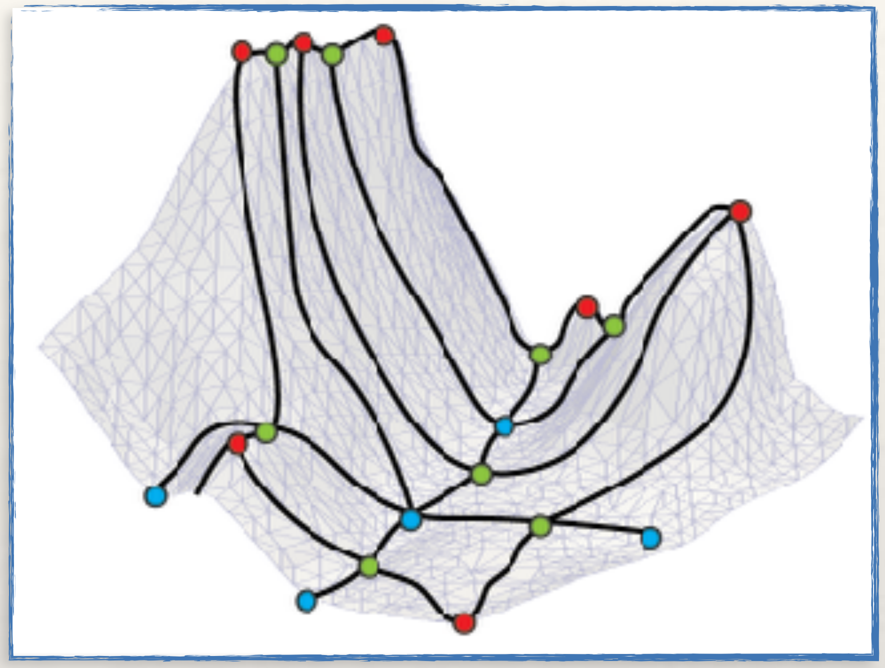
Various **discretizations** of Morse theory:

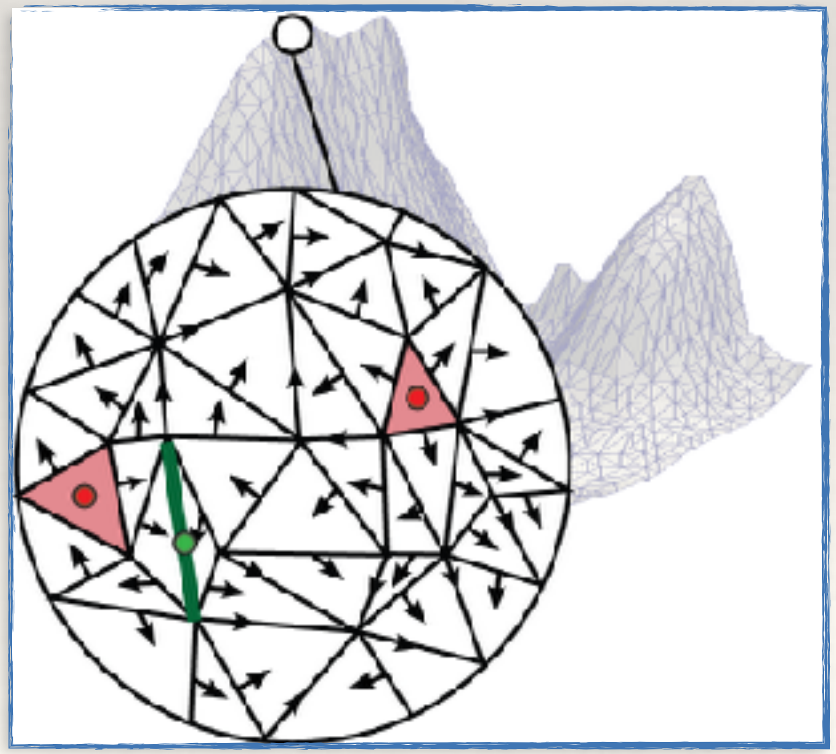
- *Piecewise linear Morse theory* [Banchoff '67]
- *Watershed transform* [Meyer '94]
- ***Discrete Morse theory*** [Forman '98]

Discrete Morse Theory

Combinatorial counterpart of Morse theory:

- Introduced for **cell complexes**
- Gives a compact **homology-equivalent** model for a shape
- **Derivative free** tool for computing segmentations of shapes





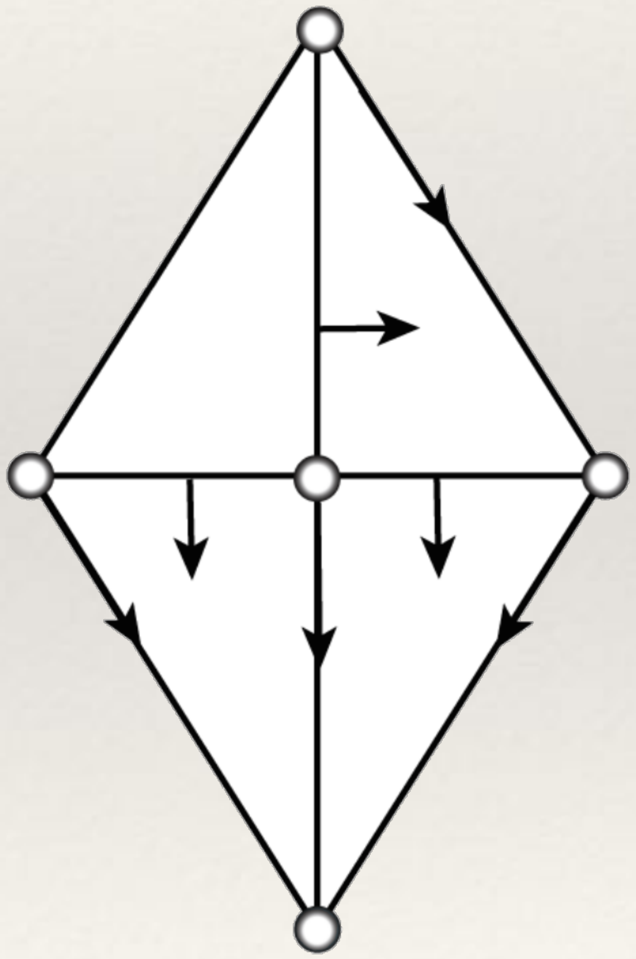
Discrete Morse Theory

Discrete Morse Theory:

Gradient of a function is **simulated by a matching** V of the simplices in Σ

A **matching** V is a **collection of pairs** (σ,τ) such that:

- σ, τ are **incident** simplices of dimension k and $k+1$
- each simplex of Σ is **in at most one pair** of V



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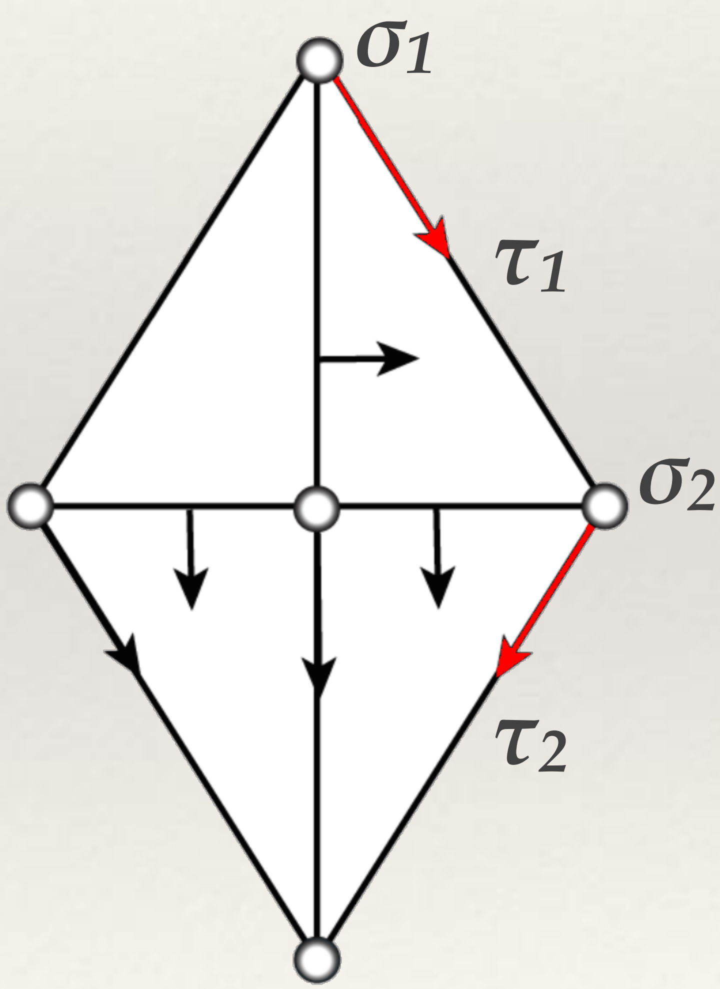


V-path: Sequence of pairs of V

$$(\sigma_1,\tau_1), (\sigma_2,\tau_2), \ldots, (\sigma_{r-1},\tau_{r-1}), (\sigma_r,\tau_r)$$

such that

- σ_{i+1} is a k-simplex face of the $(k+1)$-simplex τ_i
- σ_{i+1} is different from σ_i

A matching V is called **Forman gradient** if it is **free of closed V-paths**

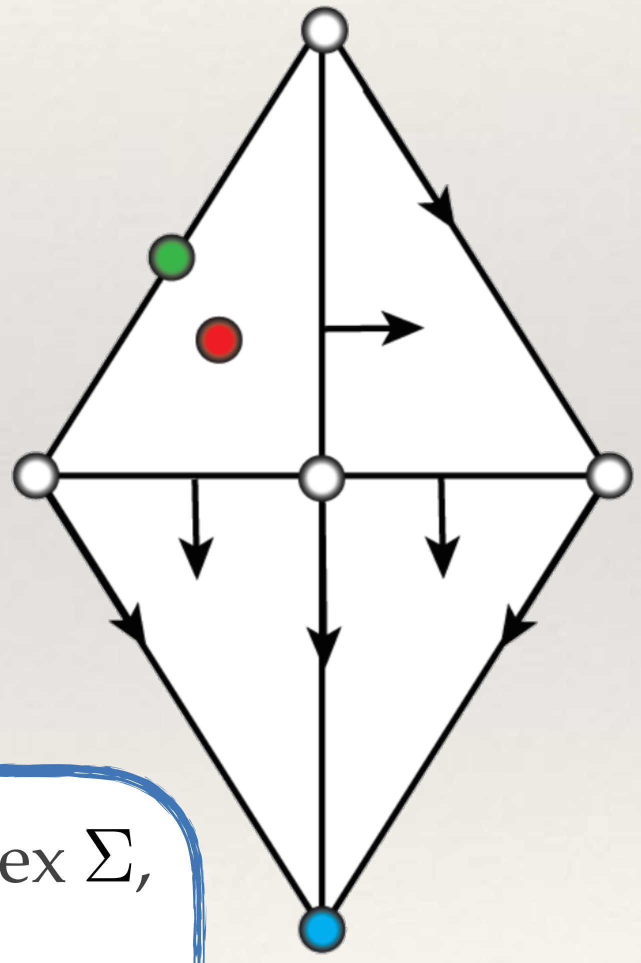
Discrete Morse Theory

Unpaired simplices of dimension k are denoted as
critical simplices of index k

Discrete Morse Complex:

A chain complex whose:

- k-cells $\longleftrightarrow$ critical simplices of index k
- boundary relations are induced by V-paths



Theorem.

Given a Forman gradient V defined on a simplicial complex Σ,
the associated **discrete Morse complex is**
homologically equivalent to Σ

Discrete Morse Theory

Filtered Forman Gradient:

Given a filtration F of a simplicial complex Σ,
a Forman gradient V is a ***filtered Forman gradient of F*** if, for each pair $(\sigma,\tau) \in V$,
there exists p such that $\sigma,\tau \in \Sigma^p$ and $\sigma,\tau \notin \Sigma^{p-1}$

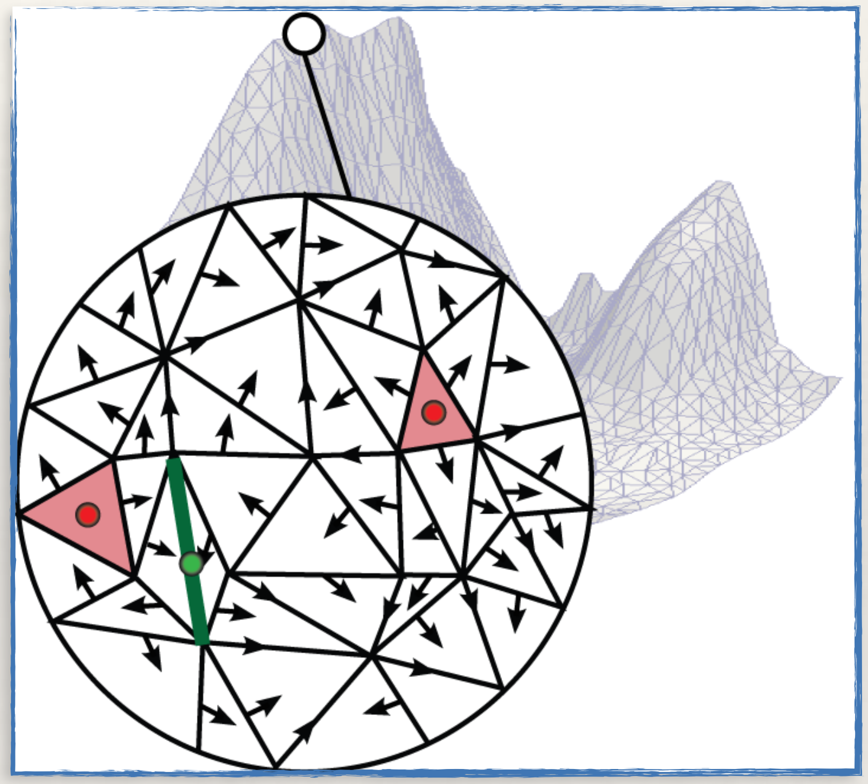
Filtration F ***naturally induces*** a filtration on the discrete Morse Complex

Theorem.

If V is a filtered Forman gradient of F, then Σ and the associated discrete Morse complex have **isomorphic persistent homology**

Discrete Morse Theory

Let Σ be a simplicial complex of dimension d



Discrete Morse Theory

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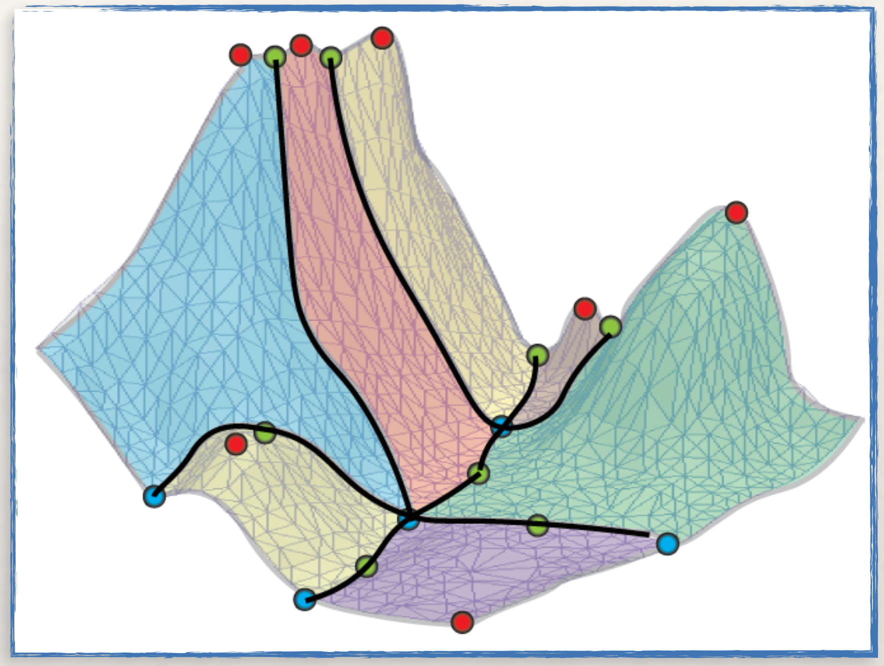
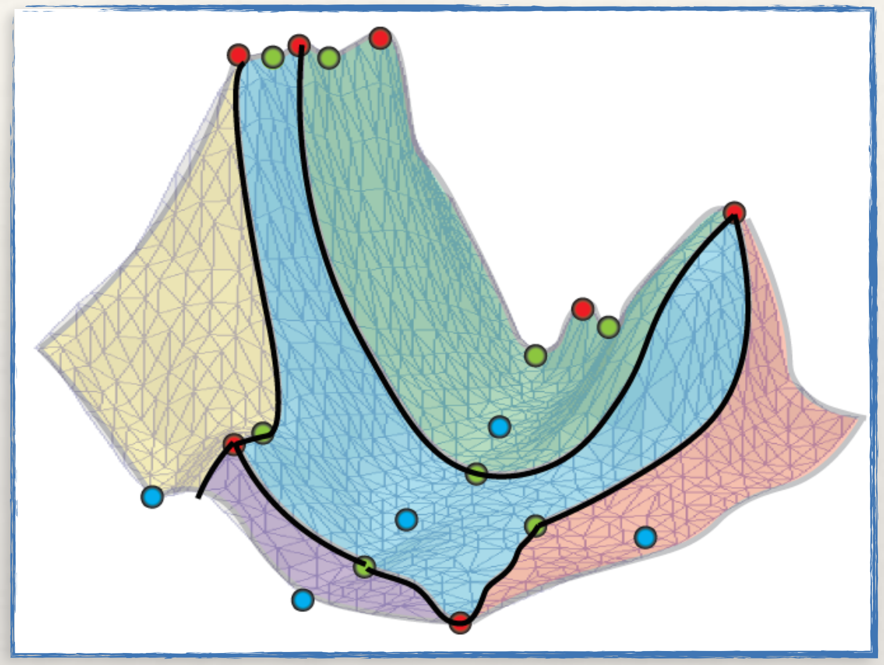


Navigating the V-paths, one can retrieve:

- **Descending Morse complex Γ_D**
  - generated by collection of the d-cells representing the regions of influence of the ***maxima*** of f: **k-cells of $\Gamma_D \longleftrightarrow$ critical simplices of index k**

Discrete Morse Theory

Let Σ be a simplicial complex of dimension d

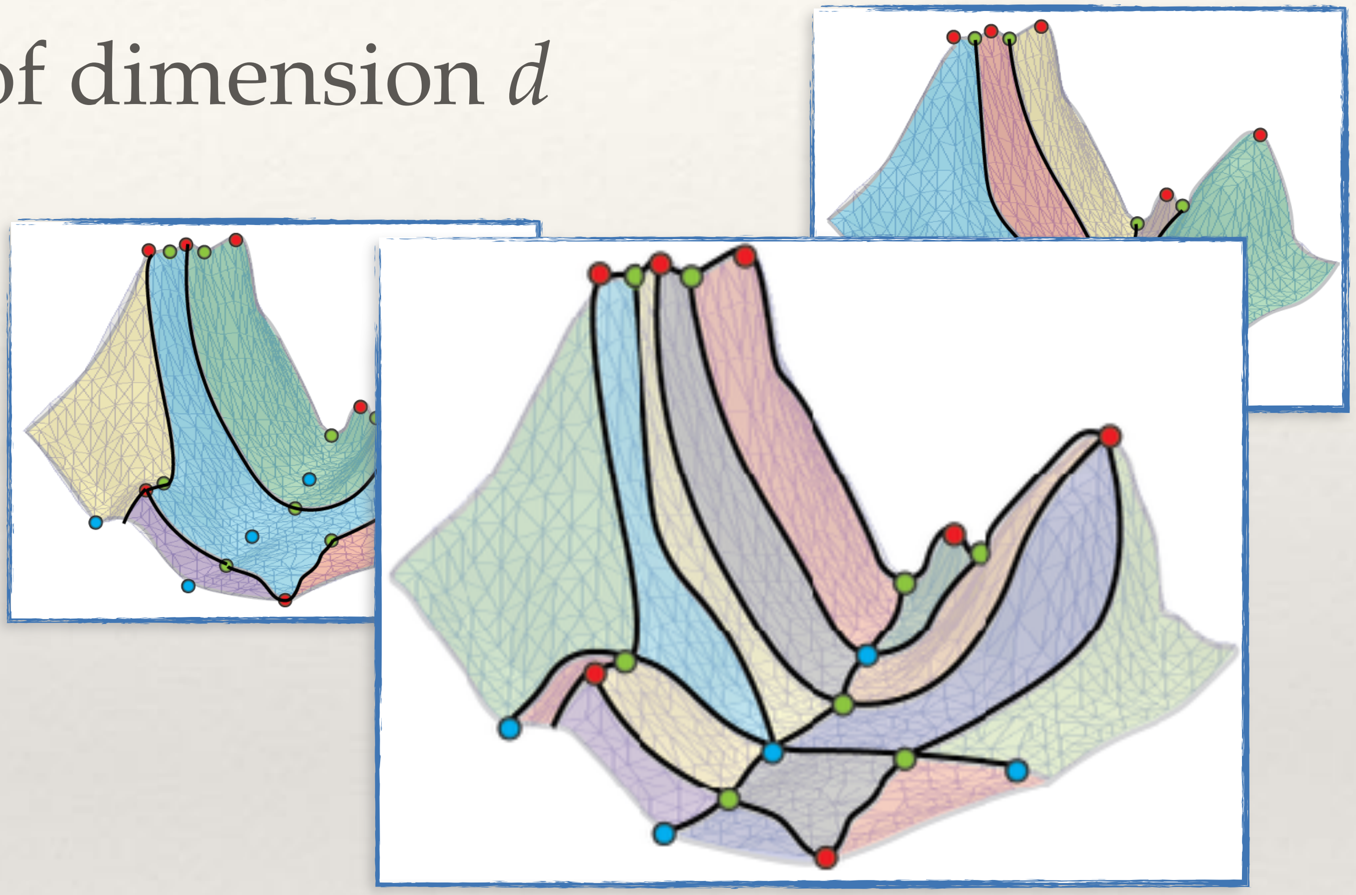


Navigating the V-paths, one can retrieve:

- **Ascending Morse complex Γ_A**
  - generated by collection of the d-cells representing the regions of influence of the ***minima*** of f : **(d-k)-cells of $\Gamma_A \longleftrightarrow$ critical simplices of index k**

Discrete Morse Theory

Let Σ be a simplicial complex of dimension d



Navigating the V-paths, one can retrieve:

- **Morse-Smale complex Γ_{MS}**
    - generated by the connected components of the ***intersection*** of the cells of the descending and ascending Morse complexes

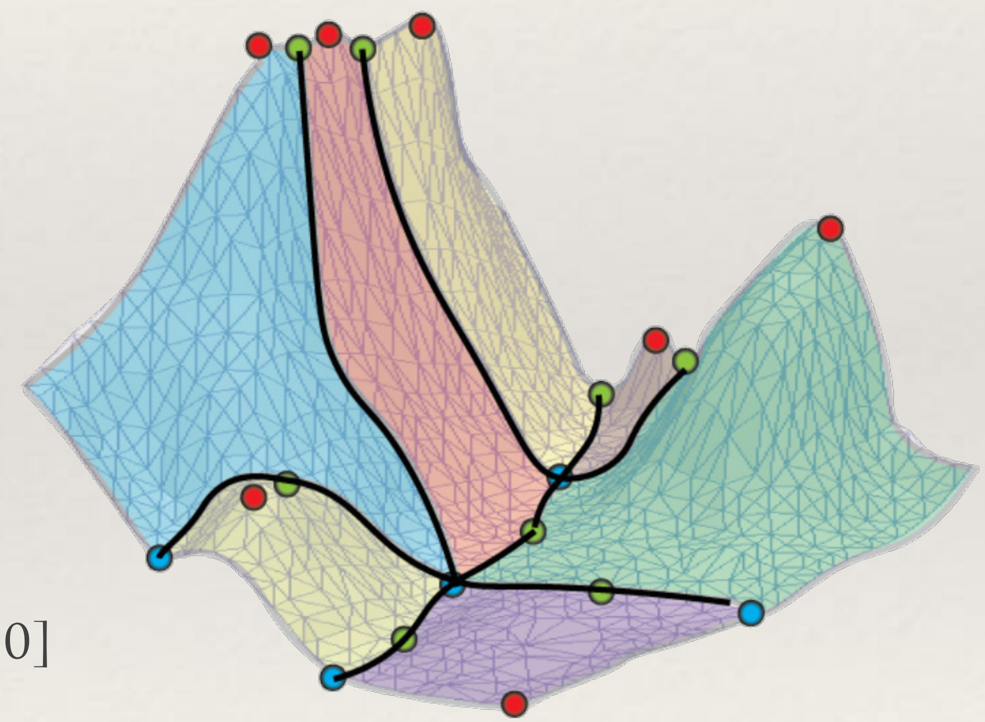
Morse Theory

Algorithms for computing Morse complexes:

Boundary-based
- ***Triangle meshes*** [Takahashi et al. '95; Edelsbrunner et al. '01; Bremer et al. '04]
- ***Tetrahedral meshes*** [Edelsbrunner et al. '03]
- ***Regular grids*** [Bajaj et al. '98; Schneider '04; Schneider '05]

Region-growing
- ***Adding triangles*** [Magillo et al. '99; Danovaro et al. '03]
- ***Adding vertices*** [Gyulassy et al. '07]

Watershed
- ***Topographic distance*** [Meyer et al. '90; Meyer '94]
- ***Simulated immersion*** [Vincent et al. '91; Soille '04]
- ***Rain falling simulation*** [Mangan et al. '99; Stove et al. '00]



Forman-based
- ***Constrained approaches*** [Cazals et al. '03; King et al. '05; Gyulassy et al. '08; Robins et al. '11; Gyulassy et al. '12]
- ***Unconstrained approaches*** [Lewiner et al. '03; Benedetti et al. '14; Harker et al. '14]
- ***Gradient traversal*** [Gunther et al. '12; Shivashankarar et al. '12; Weiss et al. '13]

Discrete Morse Theory

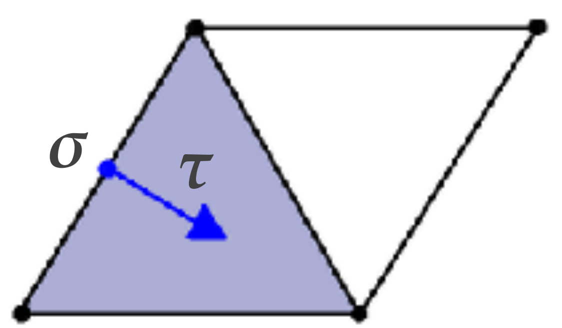
A (filtered) Forman gradient can be build by using
the **homology-preserving** operators of **reduction** and **coreduction**

Reduction and Coreduction Operators:

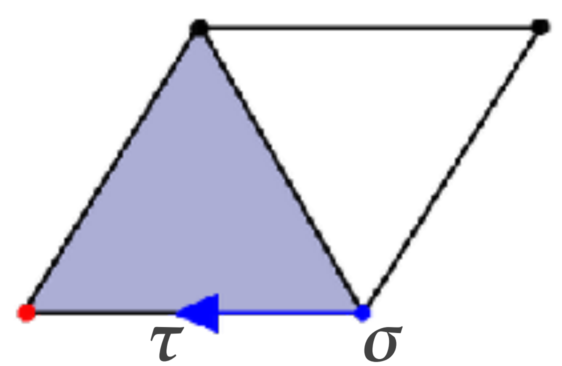
Let σ,τ be two incident simplices of dimension k and $k+1$, respectively

Pair **(σ,τ)** is called:

Reduction if
immediate coboundary of $\sigma = \{\tau\}$



Coreduction if
immediate boundary of $\tau = \{\sigma\}$



Discrete Morse Theory

Gradient through Reductions: [Benedetti et al. 2014]

Input: Σ simplicial complex
Output: V gradient vector field, A set of critical simplices

Set $\Sigma' \leftarrow \Sigma$, $V \leftarrow \varnothing$, $A \leftarrow \varnothing$

while $\Sigma' \neq \varnothing$ do

  while Σ' admits a *reduction pair* (σ, τ) do
    $V \leftarrow V \cup \{ (\sigma, \tau) \}$
    $\Sigma' \leftarrow \Sigma' \setminus \{ \sigma, \tau \}$
  end while

  Let η be a *top simplex* in Σ'
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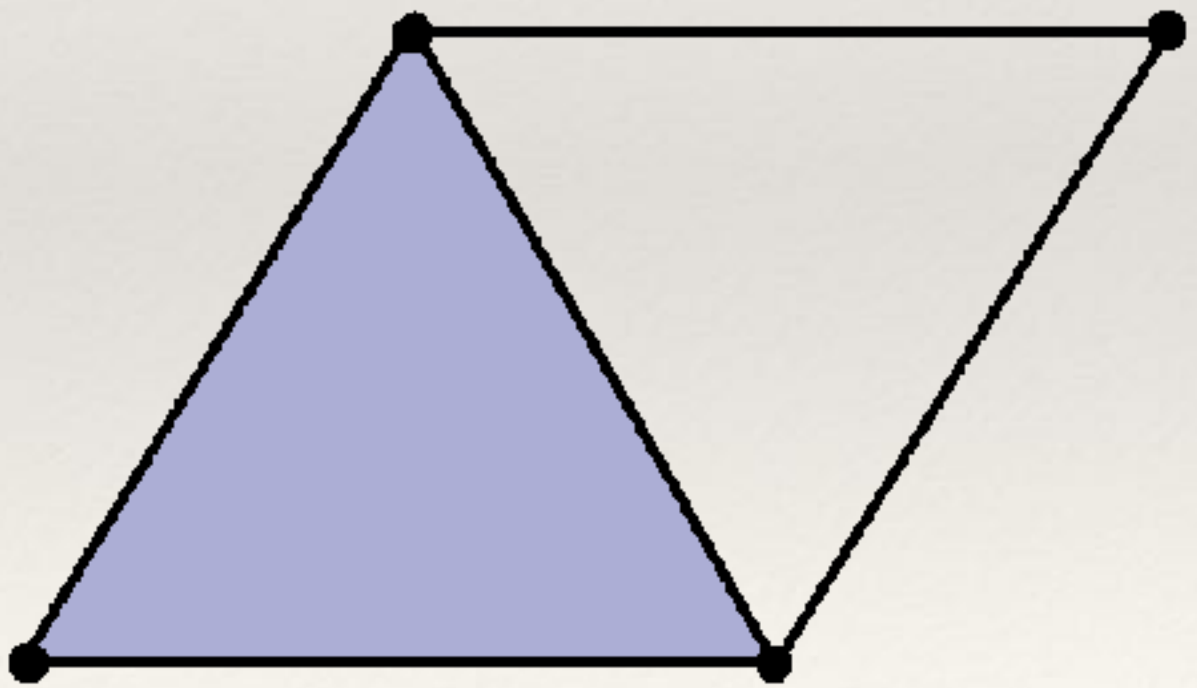

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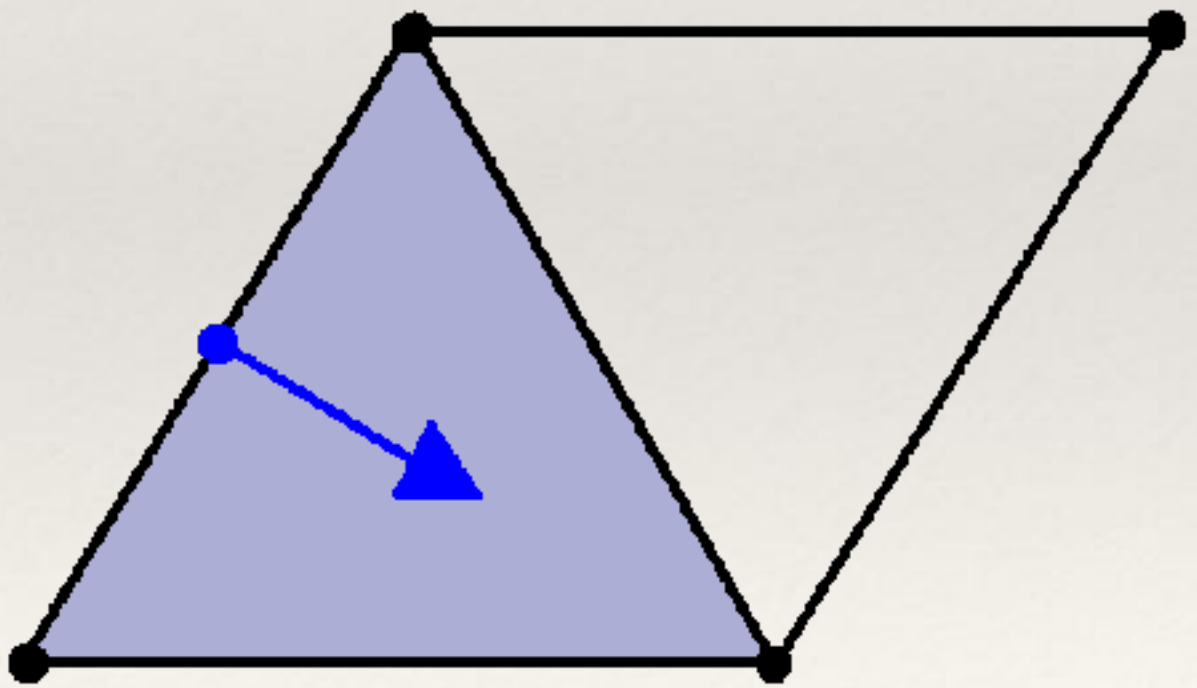

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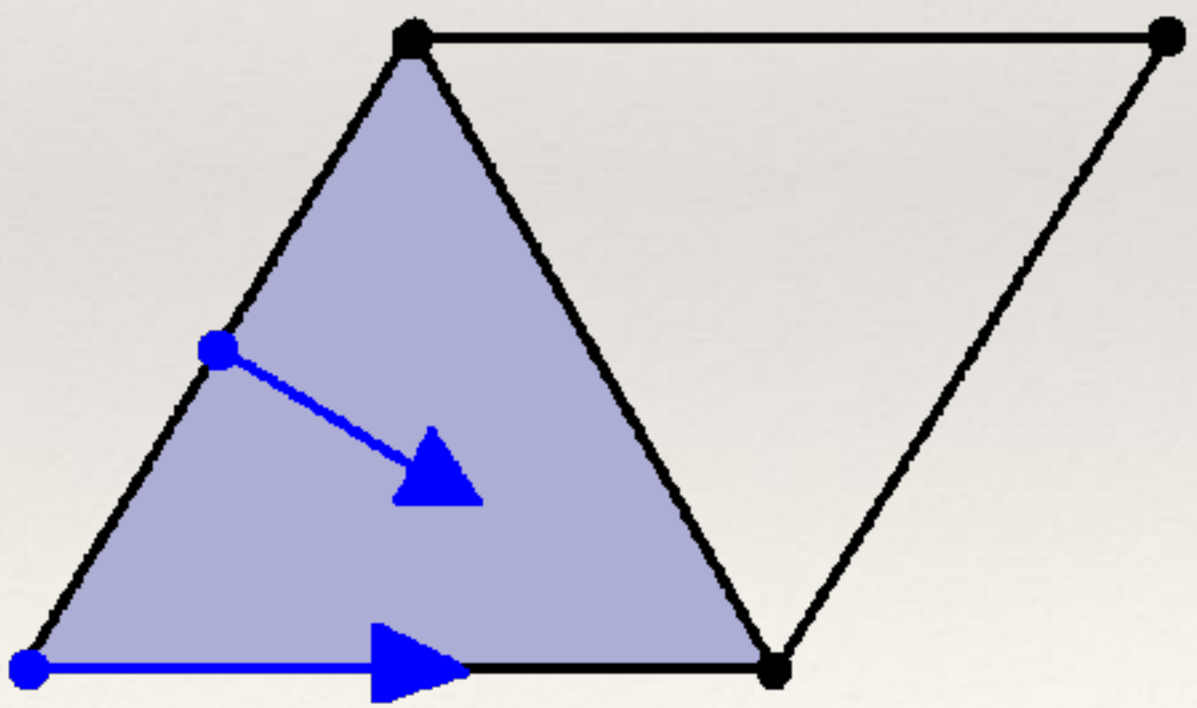

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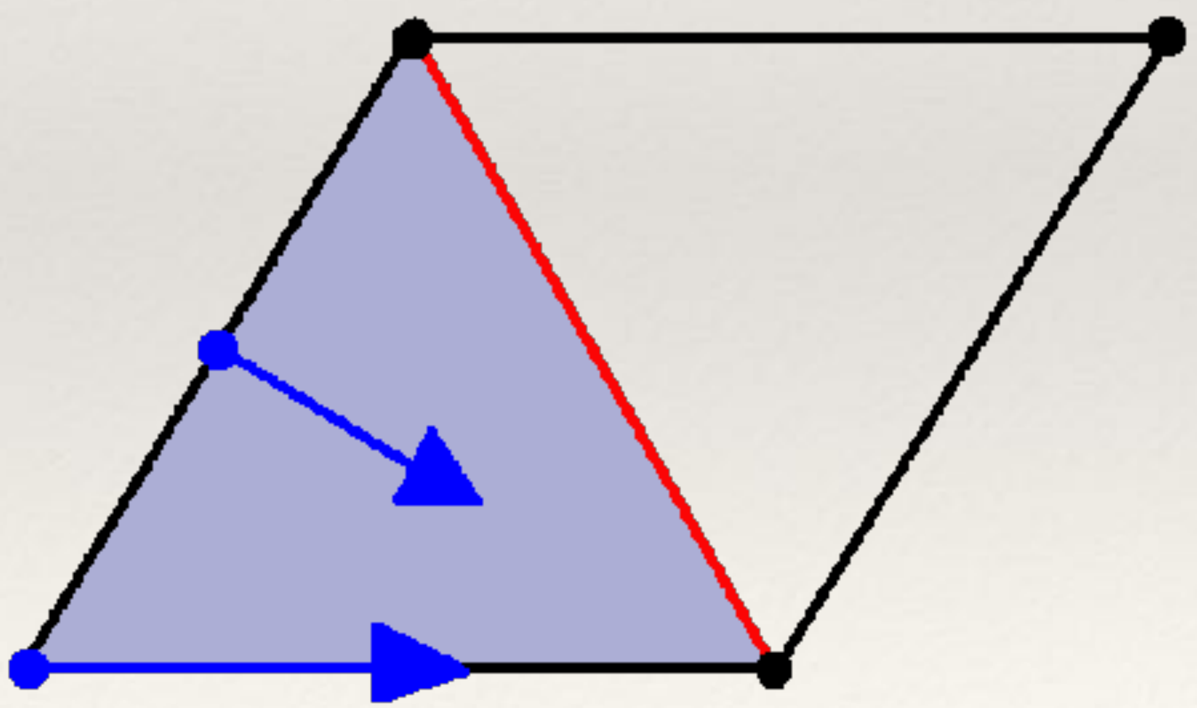

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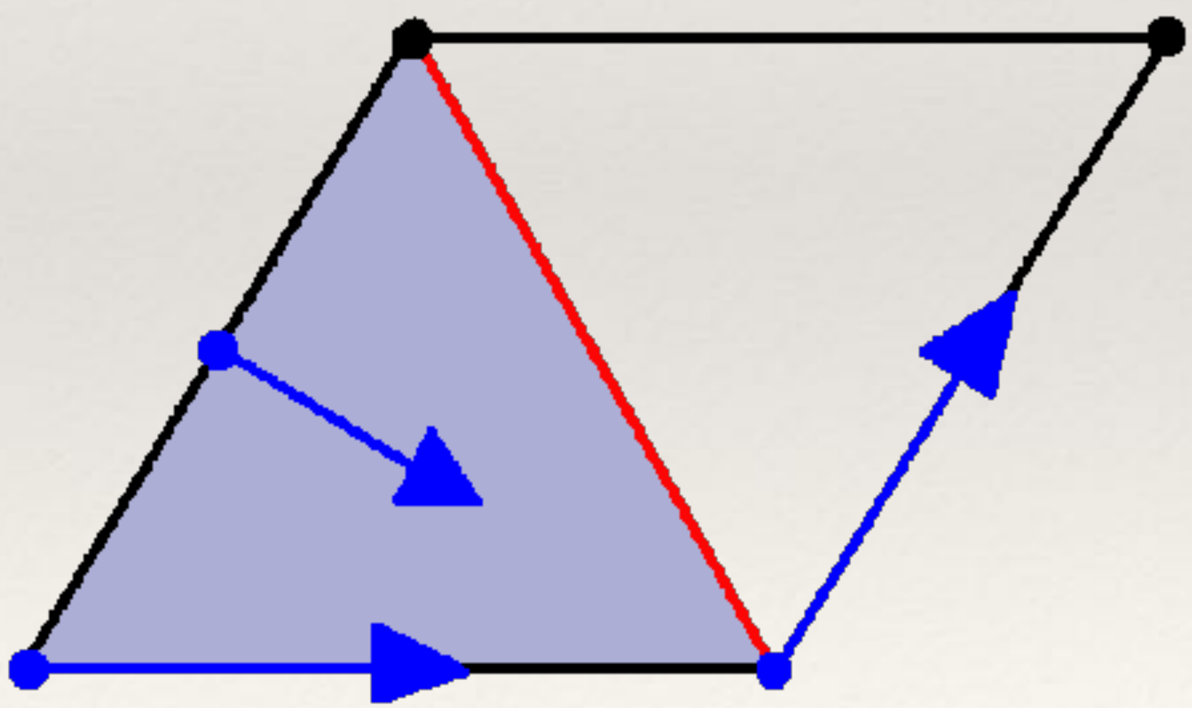

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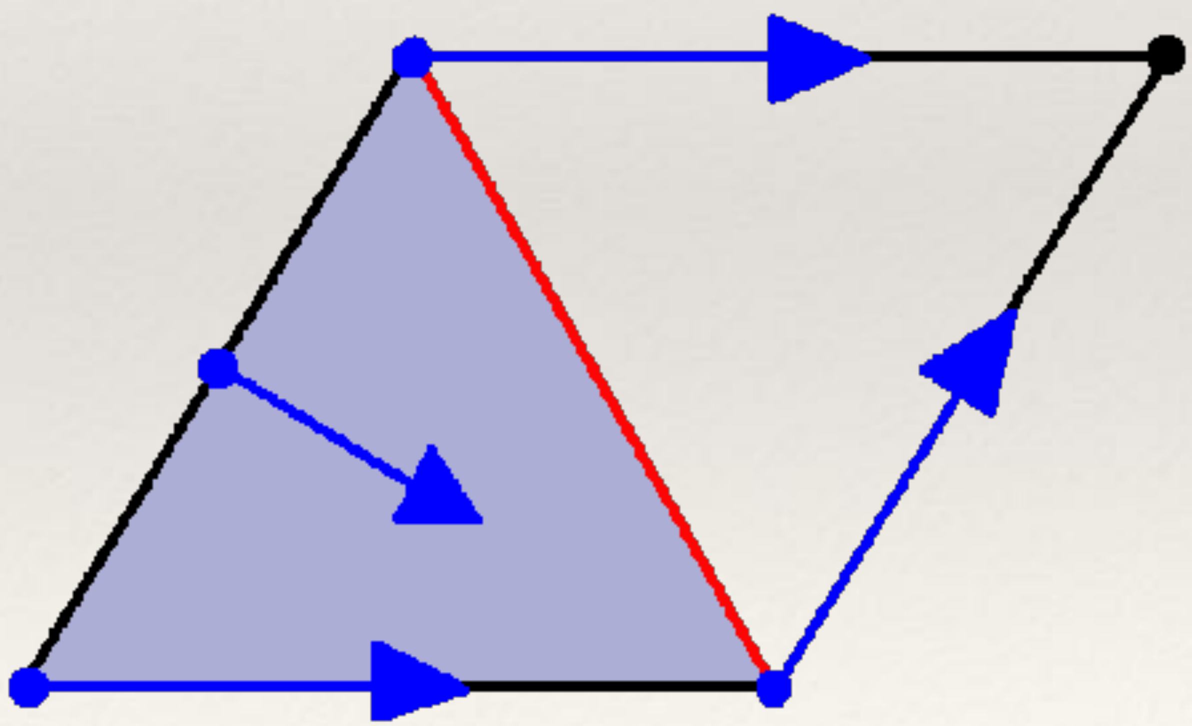

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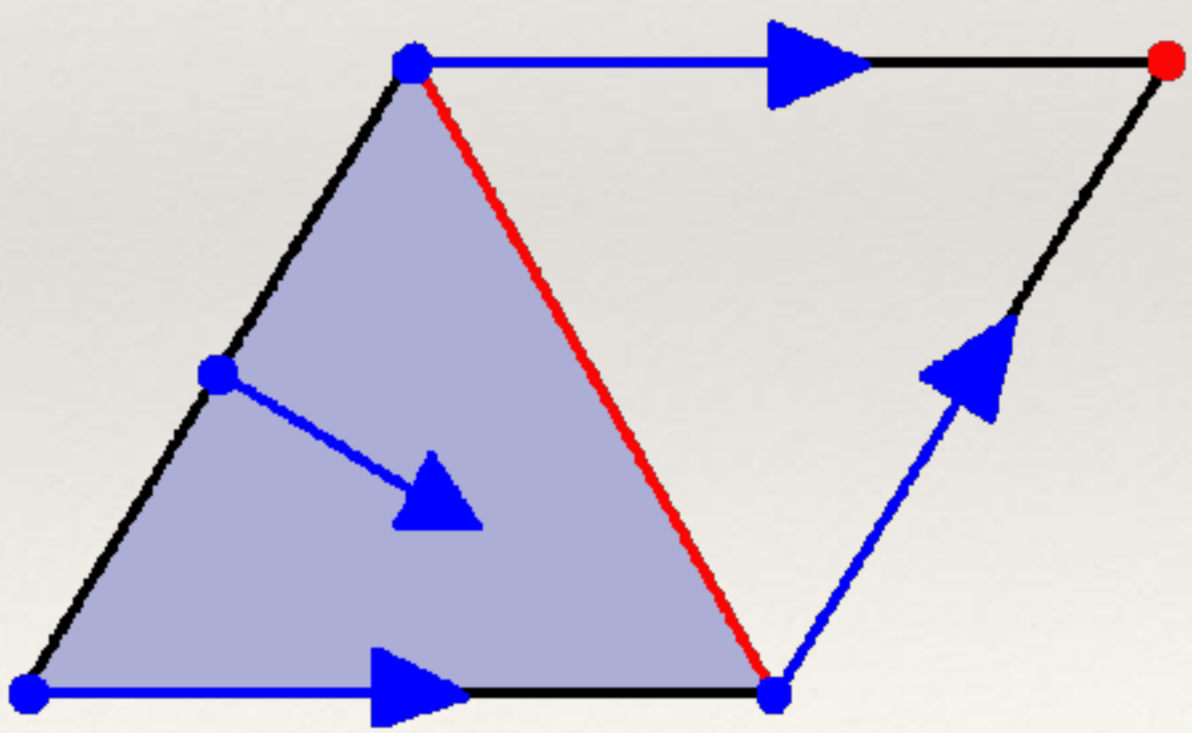

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Gradient through Coreductions: [Harker et al. 2014]

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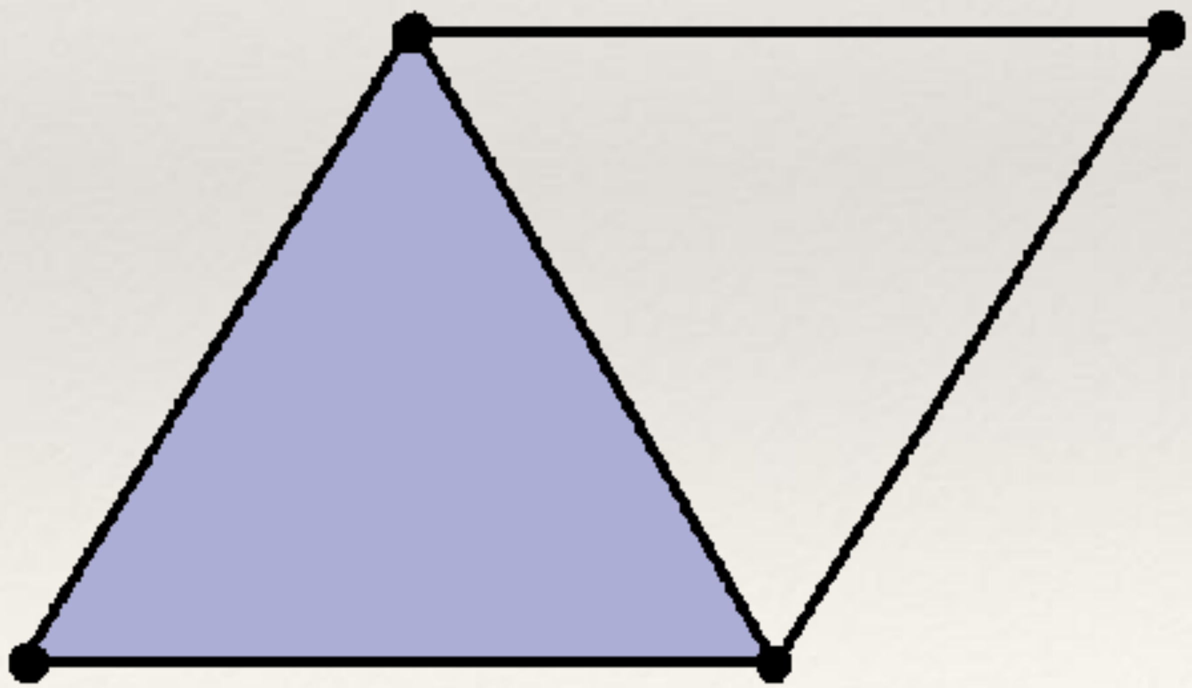

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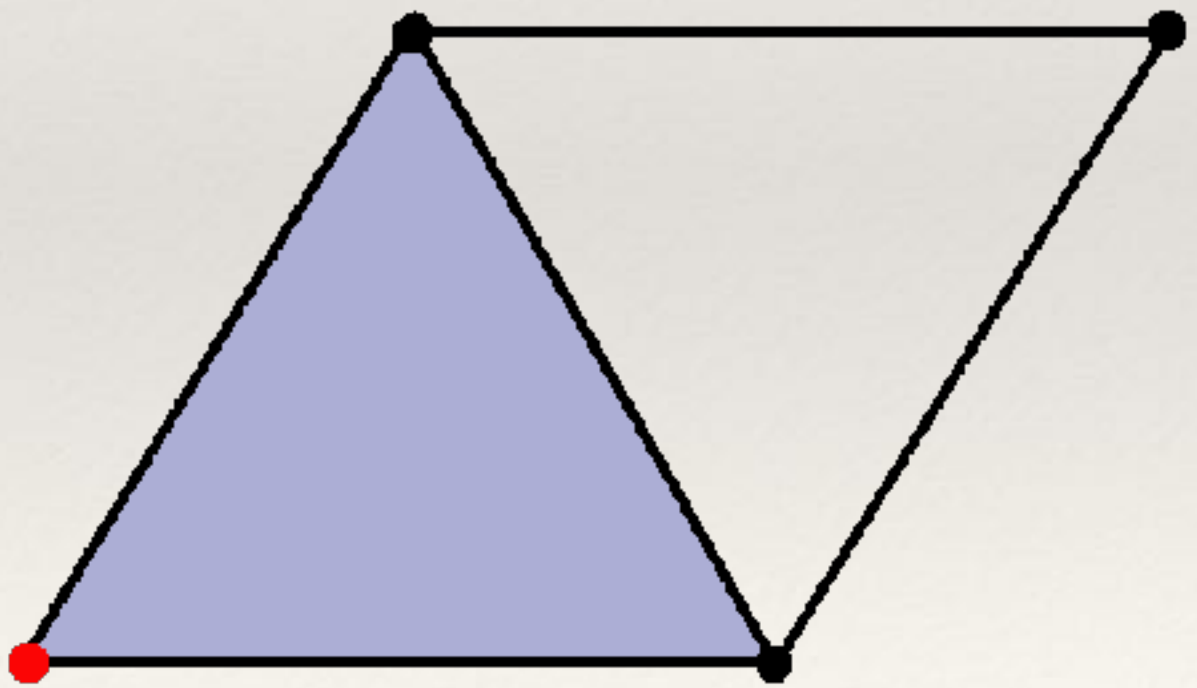

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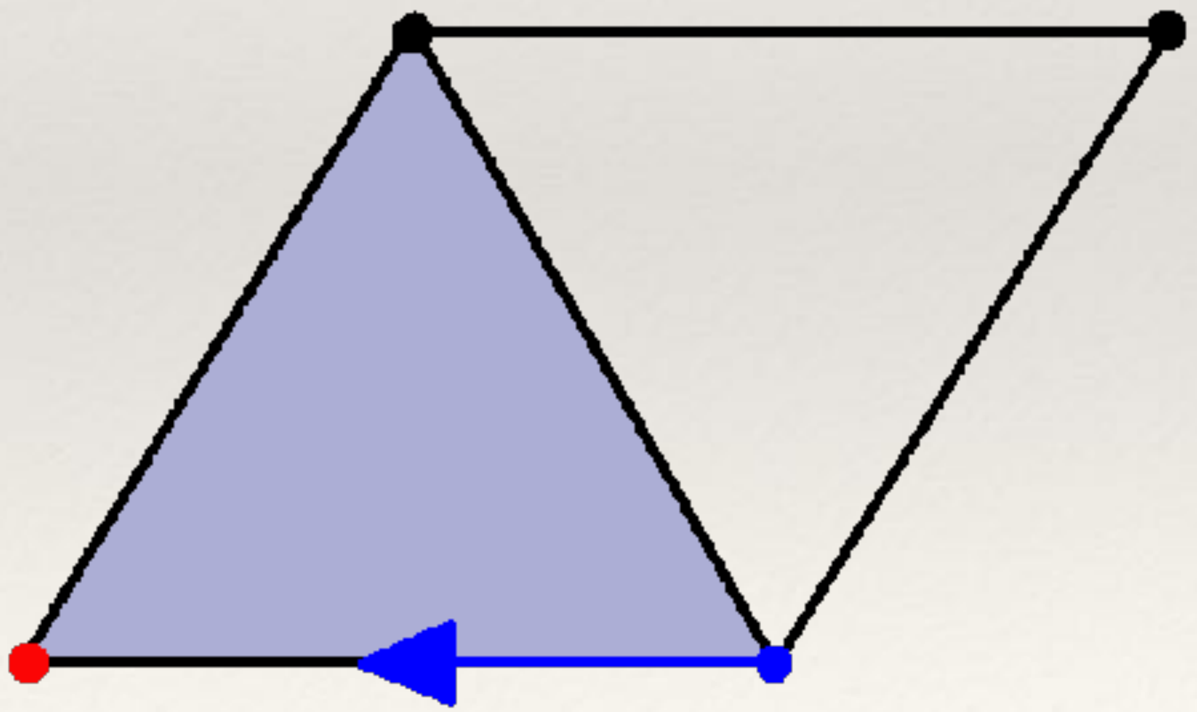

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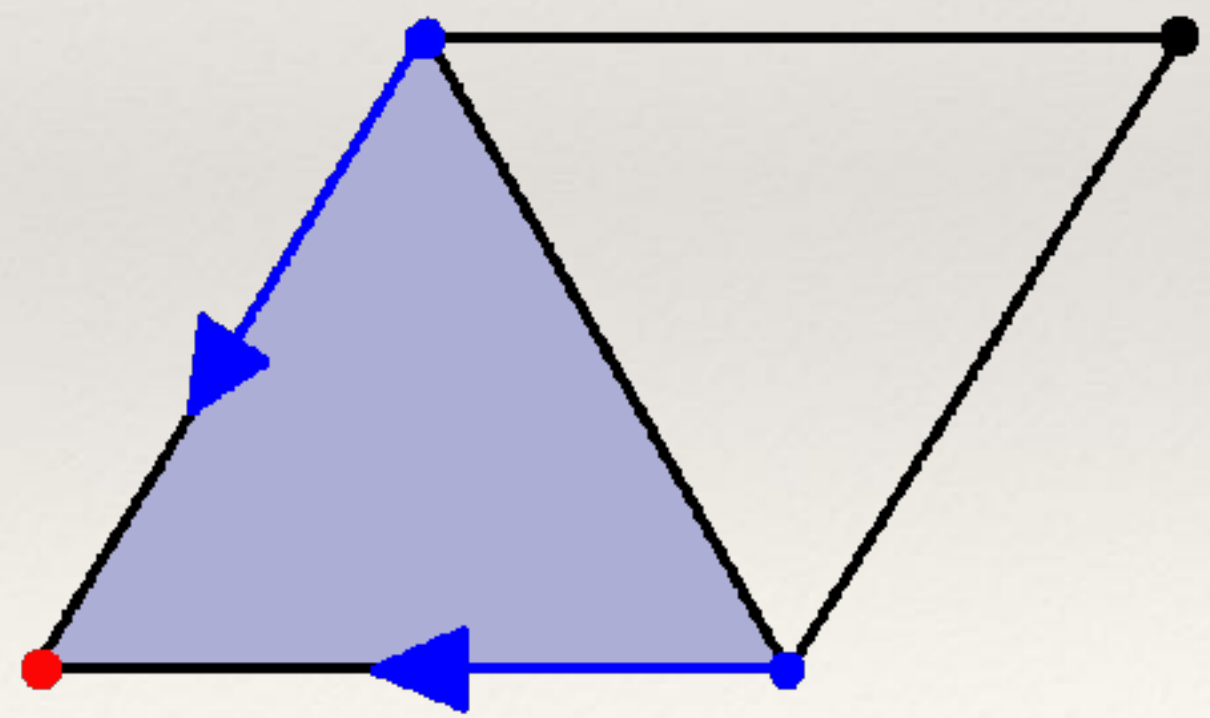

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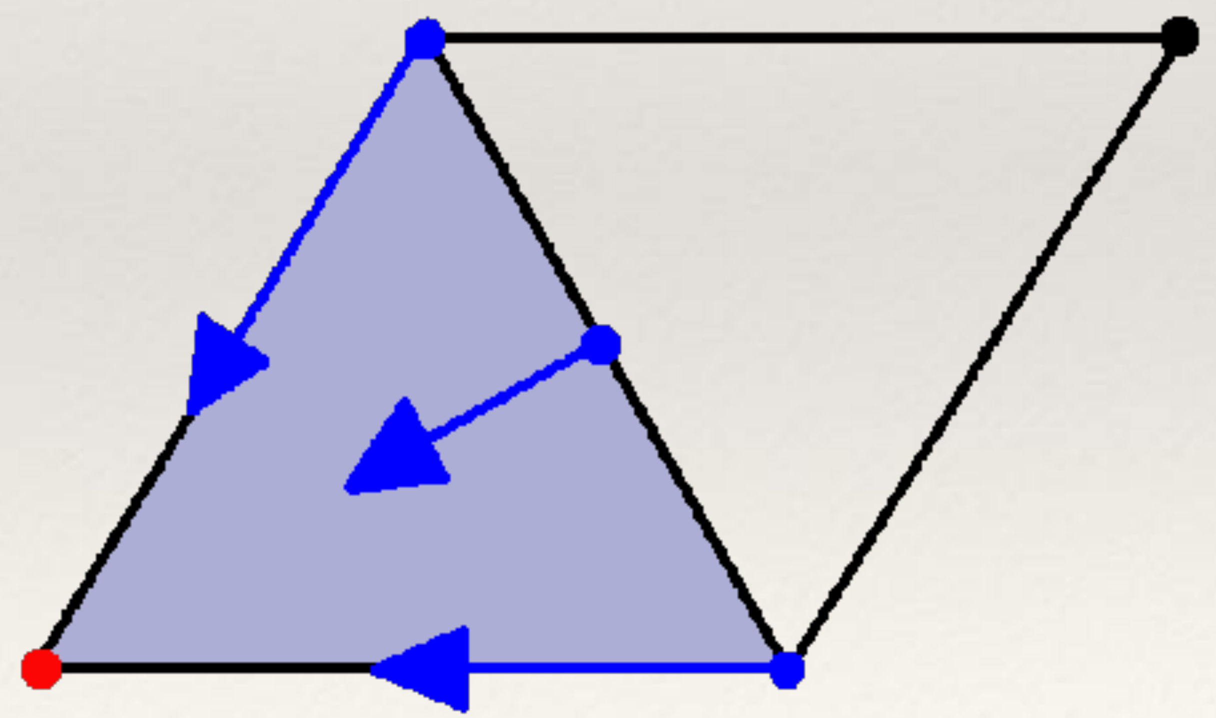

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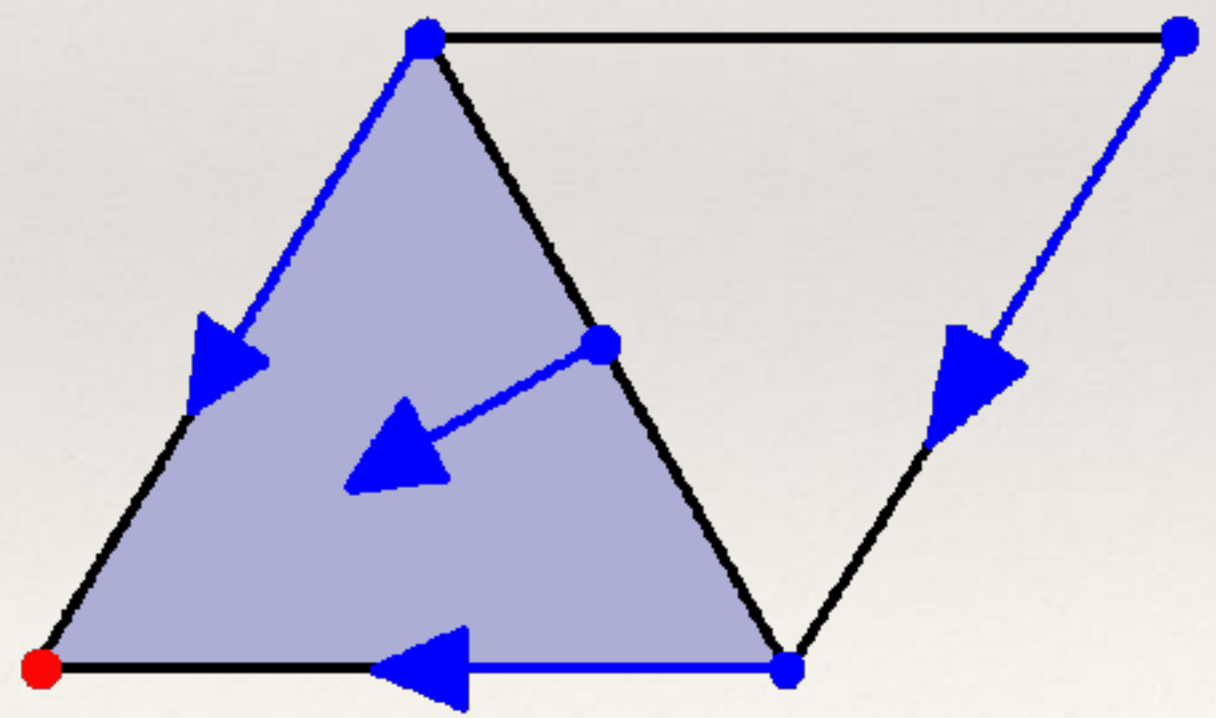

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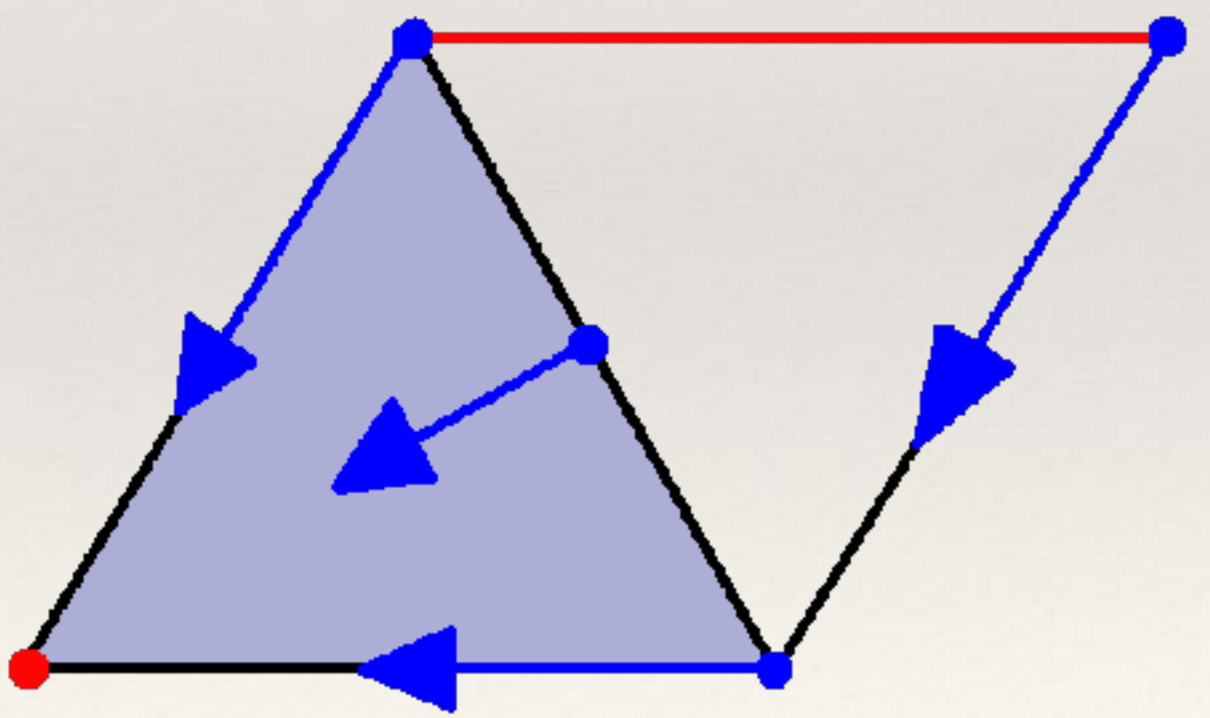


Discrete Morse Theory

Proposition.

Both the algorithms **produce** a **Forman gradient** on Σ

Which approach is able to compute a Forman gradient with **less critical simplices**?

Discrete Morse Theory

Proposition.

Both the algorithms **produce** a **Forman gradient** on Σ

Which approach is able to compute a Forman gradient with **less critical simplices**?

Reduction-based and coreduction-based approaches are **equivalent**

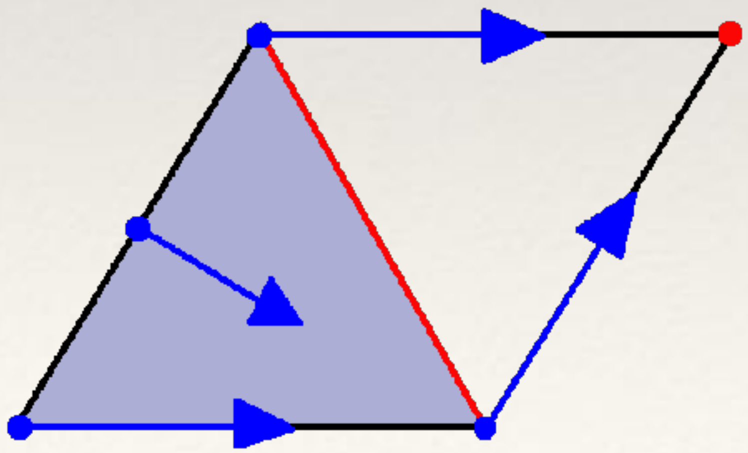
Theorem.

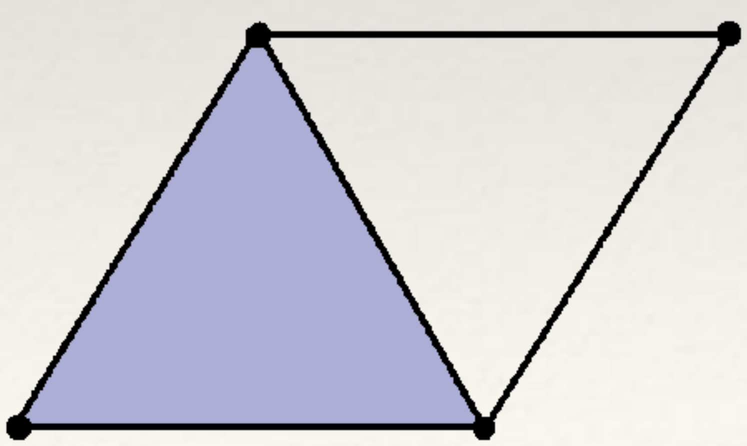
Any Forman gradient V on Σ produced by a reduction-based algorithm can be obtained through a coreduction-based algorithm; and the converse is also true

Discrete Morse Theory

Proof's guidelines: (from reduction-based to coreduction-based)

- Consider a simplicial complex Σ and run the reduction-based approach on it
- Take the sequence of reduction pairs and top simplex removals operated by the algorithm
- Reverse the order of the sequence: this new sequence represents for Σ a performable sequence of coreduction pairs and free simplex removals





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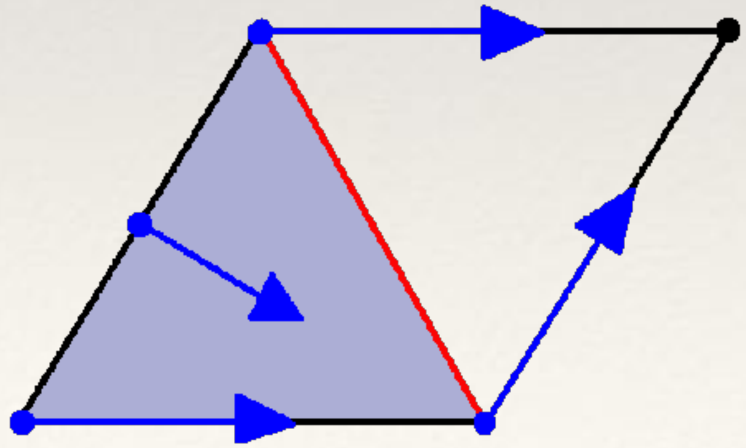

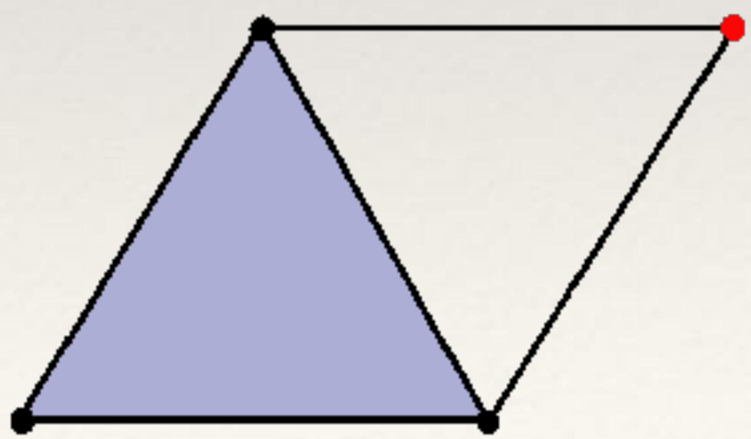

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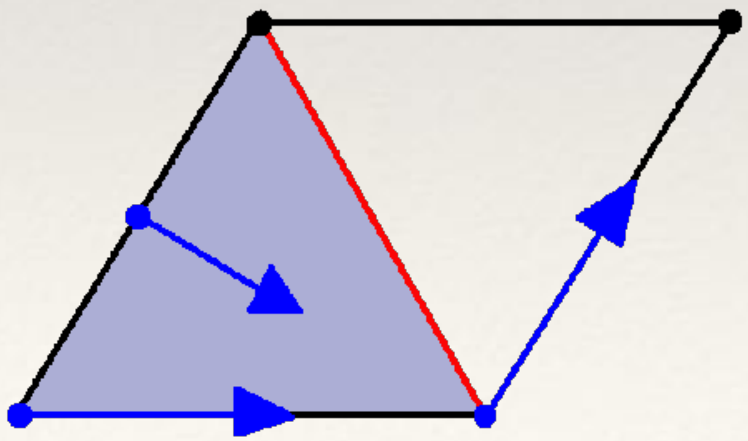

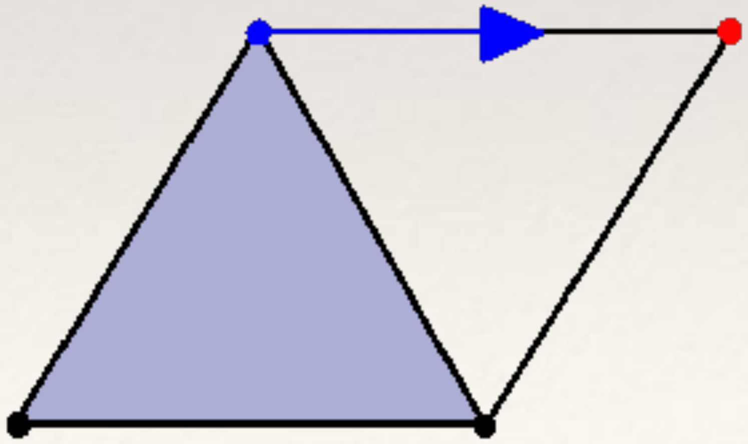

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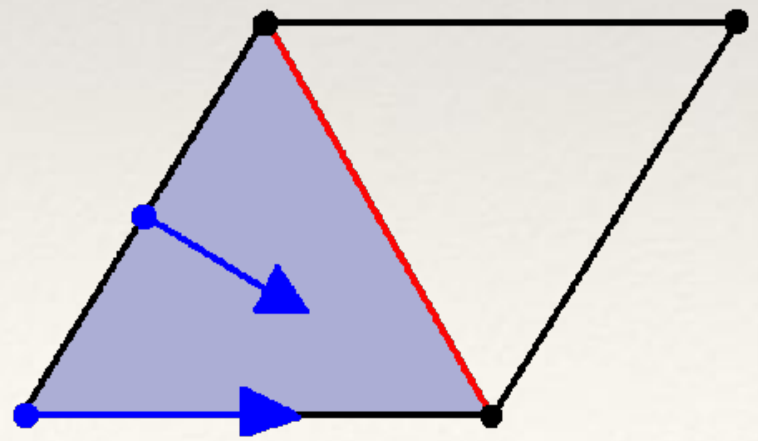

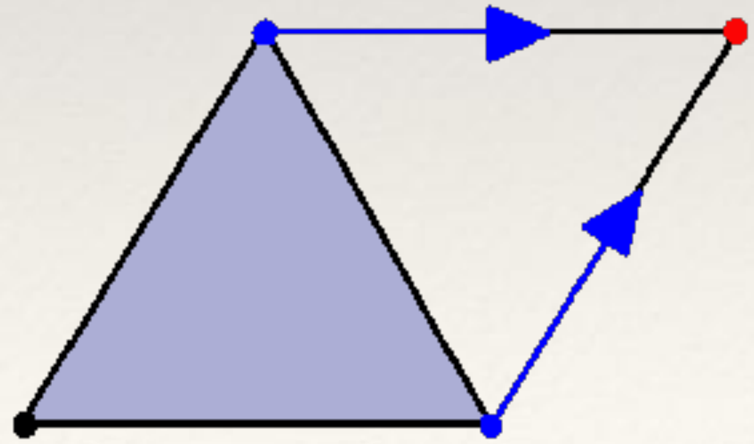

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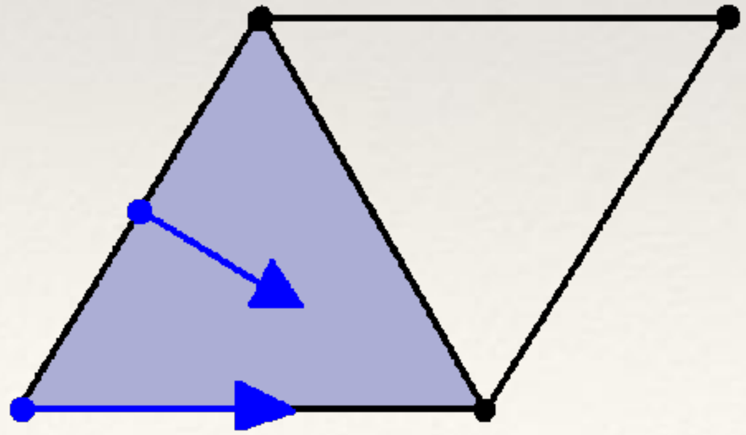

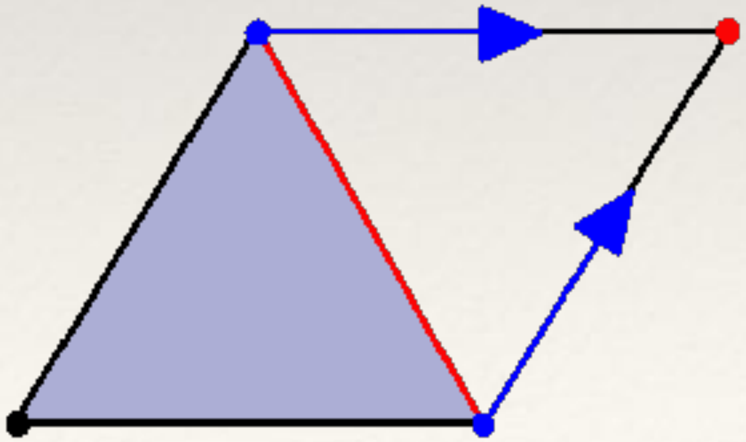

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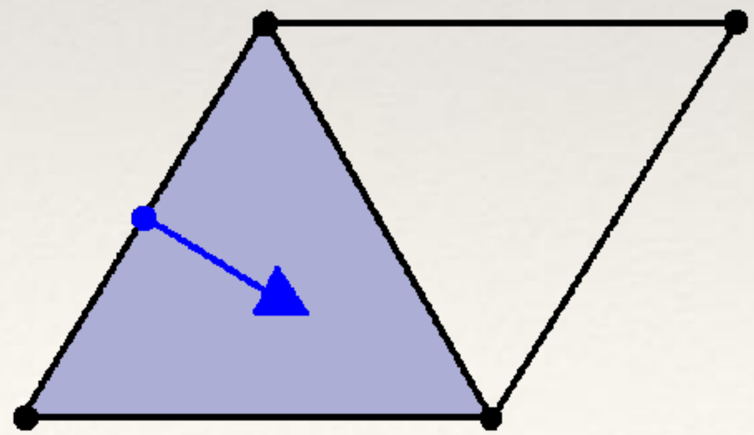

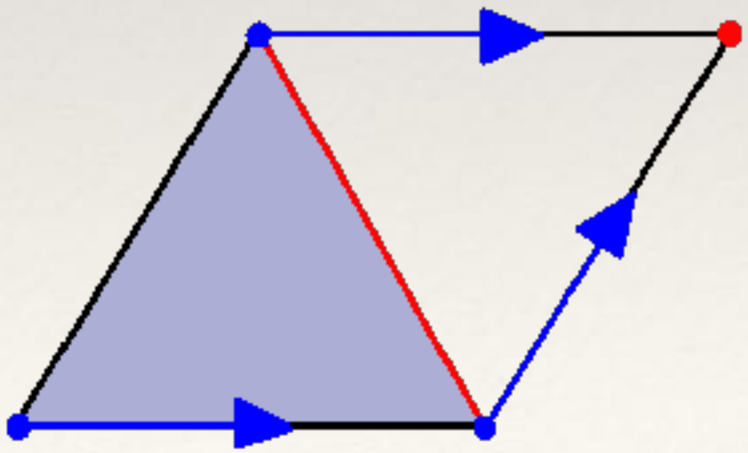

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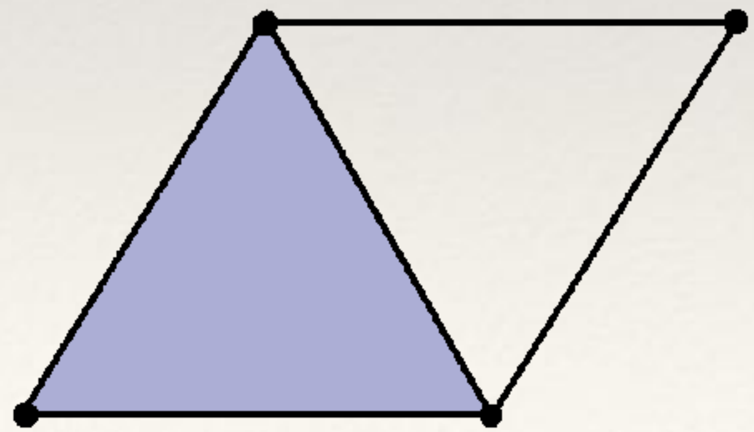

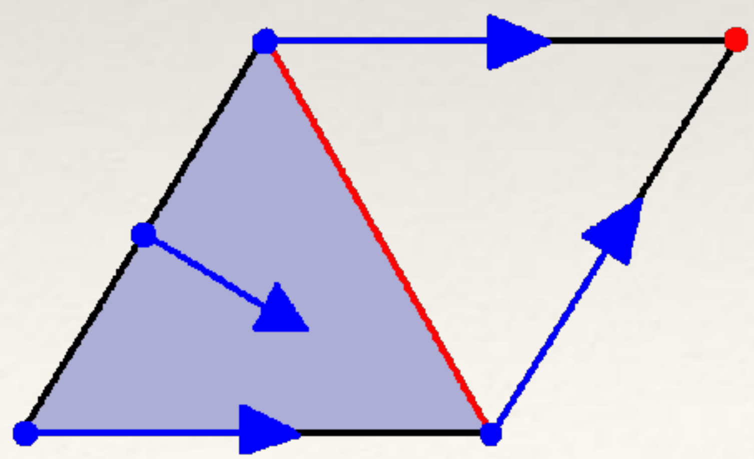

Discrete Morse Theory

Interleaved Approach:

Another class of approaches **interleaving reductions and coreductions** has been considered

Proposition.

Any interleaved approach **produces** a **Forman gradient** on Σ

Discrete Morse Theory

Interleaved Approach:

Another class of approaches **interleaving reductions and coreductions** has been considered

Proposition.

Any interleaved approach **produces** a **Forman gradient** on Σ

Each interleaved approach has **equivalent** capabilities

Theorem.

Any Forman gradient V on Σ produced by an interleaved algorithm can be obtained through a reduction-based algorithm or, equivalently, through a coreduction-based algorithm

Outline

Persistent Homology
Computation

Discrete Morse Theory and
Persistent Homology

Thank you

Ulderico Fugacci

***TU Kaiserslautern**, Dept. of Computer Science*